

Two-Copy Onset of Strict Superadditivity for Measured Relative Entropy

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Quantum relative entropy is the asymptotic state-discrimination rate and, for an isolated system initially Gibbs prepared and unitarily driven, equals the dimensionless irreversible work. We prove that for faithful (full-rank) finite-dimensional states, measured relative entropy on every $n \geq 2$ copies is additive if and only if the states commute. Noncommuting pairs gain already at two copies through an explicit two-vector rotation. For a finite-temperature Gibbs-prepared isolated drive with IID access to the driven state and final Gibbs reference, strict optimized two-copy gain is equivalent to energetic coherence. For qubits, deterministic optimization recovers stationary candidates and the explicit rotation quantifies a direct finite-depth witness. All numerical results reported here are deterministic classical calculations.

a. Introduction.— Entropy production is process dependent [1–3]. In the specific isolated protocol used here, a finite-dimensional system starts in $\pi_0 = e^{-\beta H_0}/Z_0$ and is driven unitarily to $\rho_\tau = U\pi_0 U^\dagger$. Its dimensionless irreversible work obeys [4]

$$\begin{aligned} \Sigma_{\text{iso}} &\equiv \beta(W - \Delta F) = D(\rho_\tau \| \pi_\tau), \\ D(\rho \| \sigma) &= \text{Tr } \rho(\ln \rho - \ln \sigma), \end{aligned} \quad (1)$$

where $W = \text{Tr}(H_\tau \rho_\tau) - \text{Tr}(H_0 \pi_0)$, $\Delta F = -\beta^{-1} \ln(Z_\tau/Z_0)$, $\pi_\tau = e^{-\beta H_\tau}/Z_\tau$, $\beta > 0$, and logarithms are natural. This identity follows from unitary entropy conservation. In the distinct global system–environment framework, an initially factorized system and thermal environment instead give $\Sigma_{SE} = D(\rho_{SE}(\tau) \| \rho_S(\tau) \otimes \rho_E^{\text{th}})$ [3]; the Supplemental Material distinguishes the two settings. This Letter uses only Σ_{iso} . Quantum relative entropy is monotone under physical processing and is the optimal asymptotic type-II error exponent in asymmetric quantum hypothesis testing at fixed type-I error [5–9]; related many-copy discrimination is governed by the quantum Chernoff bound [10]. Equation (1) therefore connects isolated-drive irreversibility to an operational distinguishability rate.

A concrete readout $M = (M_y)$ maps the states to classical laws $p_y = \text{Tr}(\rho M_y)$ and $q_y = \text{Tr}(\sigma M_y)$. The largest classical relative entropy available from one copy, built from the Kullback–Leibler divergence [11], is

$$D_M(\rho \| \sigma) = \sup_M \sum_y p_y \ln \frac{p_y}{q_y}. \quad (2)$$

Because every quantum experiment or processor ultimately returns classical samples, D_M is the largest expected log-likelihood ratio obtainable from a one-copy readout. It is a hardware-agnostic information ceiling, while a device restricts the optimization to its available gate set. Variational estimators for D_M have been analyzed numerically, while semidefinite and operator-optimization methods provide complementary computational routes [12–14]. Two-copy entangling measurements have been implemented for metrology and state-discrimination tasks, with photonic experiments providing

adjacent collective-readout precedents [15–17]. Phase-adjusted two-vector rotations and their projective first-variation calculus were developed in Ref. [18]. We use that calculus as a local optimization tool. The present result supplies the tensor-product cross-score construction, the logarithmic-mean tensor gap, the measured-relative-entropy derivative, and the strict onset at every $n \geq 2$. The known fixed-length one-copy-adaptive boundary motivates the finite-block question: does a joint readout yield a strict fixed-pair gain already at block size two for every faithful noncommuting pair?

Rank-one projective measurements attain the supremum for faithful states, and $D_M < D$ exactly for noncommuting pairs [20]. Collective measurements asymptotically recover D [8, 9], while restricted-measurement Stein problems quantify what remains accessible under constrained readouts [21–24]. Fixed-length protocols restricted to one-copy measurements with classical feed-forward have the measured-relative-entropy boundary established by Hayashi and developed in the sequential framework of Li, Tan, and Tomamichel [25, 26]. Prior work implies strict gain for some finite block whenever the states do not commute [27, Remark 4.27], and general superadditivity is known in broader state-set settings [28, Lemma 23]. To locate the finite-block onset, set $F_n = D_M(\rho^{\otimes n} \| \sigma^{\otimes n})$. Then $F_{n+m} \geq F_n + F_m$, and Fekete’s lemma [29] together with Hiai–Petz gives $\lim_{n \rightarrow \infty} F_n/n = D(\rho \| \sigma)$ [8]. Noncommutation therefore guarantees $F_n/n > F_1$ for some finite n , while the known asymptotic argument leaves the first strict block undetermined. Equation (3) fixes that onset at two copies.

First main result.— For faithful density operators ρ, σ on a finite-dimensional Hilbert space and each fixed integer $n \geq 2$,

$$D_M(\rho^{\otimes n} \| \sigma^{\otimes n}) = nD_M(\rho \| \sigma) \iff [\rho, \sigma] = 0. \quad (3)$$

When the states do not commute, a phase-adjusted rotation of two vectors in a product-optimal basis has a strictly positive first derivative. Thus Eq. (3) strengthens the known “some finite n ” nonadditivity [27, Remark 4.27] to every $n \geq 2$ with an explicit direction. Repeating an n -copy block over many independent blocks gives the classical Stein exponent $D_M(\rho^{\otimes n} \| \sigma^{\otimes n})/n$ per signal copy [30].

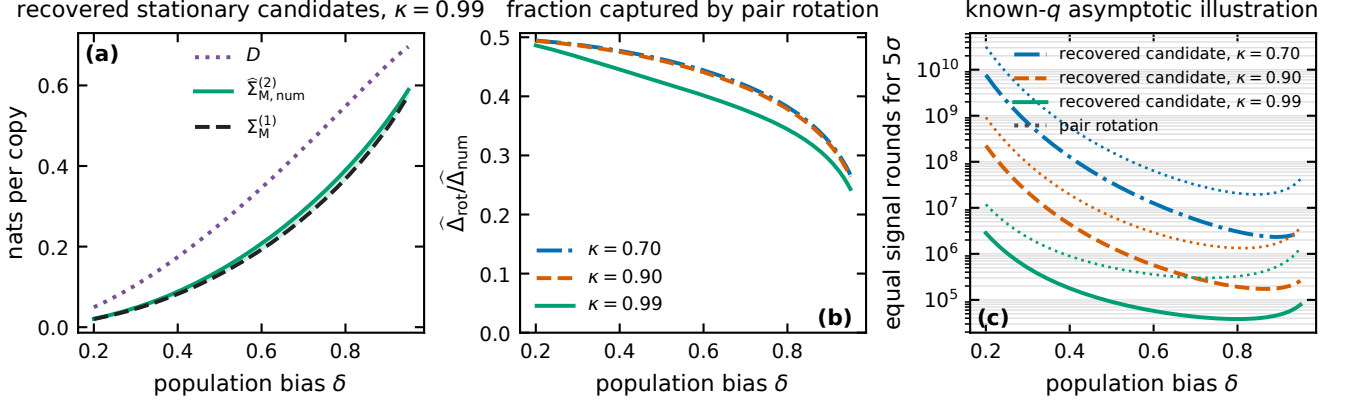


FIG. 1. Direct numerical audit of the analytic qubit family. (a) Analytic one-copy optimum, numerically recovered two-copy stationary candidate $\hat{\Sigma}_{M,num}^{(2)}$, and quantum relative entropy at $\kappa = 0.99$; shading is the recovered per-copy gap $\hat{\Delta}_{num}$. (b) Fraction $\hat{\Delta}_{rot}/\hat{\Delta}_{num}$ captured by numerical maximization within the explicit pair-rotation family. (c) Known-reference, equal-round, delta-method estimate [19] of the signal rounds needed for a nominal 5σ separation; dashed and dotted curves correspond to the recovered stationary candidate and explicit pair rotation. Equal N rounds in the one- and two-copy arms consume $3N$ signal copies; reference calibration requires additional copies. Panel (c) is an asymptotic planning calculation without a finite-sample or device-performance guarantee.

Every faithful noncommuting pair therefore improves on repeated one-copy readout already at block size two.

b. Formalism.— Berta, Fawzi, and Tomamichel prove equality of the projective and arbitrary-POVM optimizations [20, Theorem 2]; rank-one refinement and compactness therefore give a one-copy optimal rank-one PVM $P = (|j\rangle\langle j|)$ for faithful states. Set $p_j = \langle j|\rho|j\rangle$, $q_j = \langle j|\sigma|j\rangle$, and $t_j = p_j/q_j$. Faithfulness makes these quantities strictly positive, so the operator below lies in the interior of the positive cone. Without faithfulness, outcome probabilities may vanish and the optimizer need not be interior; the stationarity argument and classification are asserted only for faithful pairs. The variational expression of Berta, Fawzi, and Tomamichel [20, Lemma 1], within the broader framework of Ref. [31],

$$D_M(\rho||\sigma) = \sup_{\Omega > 0} \{ \text{Tr}(\rho \ln \Omega) + 1 - \text{Tr}(\sigma \Omega) \} \quad (4)$$

is attained at $\Omega = \sum_j t_j |j\rangle\langle j|$. Interior stationarity, $D \ln_{\Omega}(\rho) = \sigma$, gives

$$\rho_{jk} = L(t_j, t_k) \sigma_{jk}, \quad L(a, b) = \frac{a - b}{\ln a - \ln b}, \quad (5)$$

$$L(a, b) = \int_0^1 a^s b^{1-s} ds.$$

If $\sigma_{jk} = 0$ whenever $t_j \neq t_k$, then ρ and σ are block diagonal in the equal- t blocks and $\rho = t_{\alpha} \sigma$ inside each block; they commute. Hence noncommutation supplies indices j, k with $a = t_j \neq t_k = b$ and $\sigma_{jk} \neq 0$.

Following the phase-adjusted two-vector variation of Ref. [18], set $|J\rangle = |j\rangle^{\otimes n}$ and $|K\rangle = |k\rangle^{\otimes n}$ on n copies

and replace only these two vectors by

$$\begin{aligned} |J_{\theta,\varphi}\rangle &= \cos \theta |J\rangle + e^{i\varphi} \sin \theta |K\rangle, \\ |K_{\theta,\varphi}\rangle &= -e^{-i\varphi} \sin \theta |J\rangle + \cos \theta |K\rangle. \end{aligned} \quad (6)$$

All other product-basis vectors remain unchanged. The parameter θ is the mixing angle away from the product basis, while φ is the relative phase selecting the complex tangent direction. Let $D_{out}^{(n)}(\theta, \varphi)$ denote the classical relative entropy of the resulting output laws and let $D_{JK}^{(n)}$ denote its two-active-outcome contribution. For fixed φ , define the slope $s_{JK}(\varphi) \equiv \partial_{\theta} D_{JK}^{(n)}(\theta, \varphi) \big|_{\theta=0} = 2 \text{Re}\{e^{i\varphi} B_{JK}\}$, where $R = \rho^{\otimes n}$, $S = \sigma^{\otimes n}$, and $B_{JK} = \ln(t_J/t_K) R_{JK} + (t_K - t_J) S_{JK}$. Choosing $\varphi_{\star} = -\arg B_{JK}$ gives $s_{JK}(\varphi_{\star}) = 2|B_{JK}|$ and the product readout the initial slope

$$\begin{aligned} \max_{\varphi} \partial_{\theta} D_{out}^{(n)}(\theta, \varphi) \big|_{\theta=0} \\ = 2n \left| \ln \frac{a}{b} \right| \\ \times [L(a^n, b^n) - L(a, b)^n] |\sigma_{jk}|^n > 0. \end{aligned} \quad (7)$$

For $a \neq b$, the function $x(s) = a^s b^{1-s}$ is nonconstant. Strict Jensen convexity of $x \mapsto x^n$ for $n \geq 2$ therefore gives $L(a^n, b^n) = \int_0^1 x(s)^n ds > [\int_0^1 x(s) ds]^n = L(a, b)^n$. At $\theta = 0$ the product measurement has value $n D_M(\rho||\sigma)$, proving the strict direction. Conversely, commuting states are measured without loss in a common eigenbasis, and additivity of D gives equality. The Supplemental Material records the full pair derivative and the $n = 2$ specialization [32].

The comparison class is the known fixed-length one-copy-adaptive setting [25, 26]. In the present notation,

if round r applies an arbitrary POVM to one fresh copy, selected from previous classical outcomes and using only classical inter-round memory, the chain rule gives

$$D_{\text{seq}}^{(n)} = \sum_{r=1}^n \mathbb{E}_{\rho} D\left(P_{\rho}^{Y_r|Y^{r-1}} \| P_{\sigma}^{Y_r|Y^{r-1}}\right) \leq n D_{\text{M}}(\rho \| \sigma). \quad (8)$$

Repeating a one-copy optimizer attains the bound. Equation (8) is the deterministic-horizon transcript specialization of the established sequential bound [26, Lemma 16(i)]. Separable and LOCC POVMs form broader restricted-measurement classes treated elsewhere [21–23].

c. Applications.— Write the spectral decomposition $H_{\tau} = \sum_{\varepsilon} \varepsilon \Pi_{\varepsilon}$ and define the energy pinching

$$\Delta_{\tau}(X) = \sum_{\varepsilon} \Pi_{\varepsilon} X \Pi_{\varepsilon}. \quad (9)$$

This map preserves coherence within degenerate energy eigenspaces. The standard population–coherence split [33–39] is

$$\Sigma_{\text{iso}} = D(\Delta_{\tau} \rho_{\tau} \| \pi_{\tau}) + D(\rho_{\tau} \| \Delta_{\tau} \rho_{\tau}) \equiv \Sigma_{\text{pop}} + C_{\tau}. \quad (10)$$

Define the globally optimized ensemble quantity per copy, $\Sigma_{\text{M}}^{(n)} = D_{\text{M}}(\rho_{\tau}^{\otimes n} \| \pi_{\tau}^{\otimes n})/n$. For the faithful IID state pair generated by the isolated unitary drive, $0 < \beta < \infty$ gives $\pi_0, \pi_{\tau} > 0$ and hence $\rho_{\tau} = U \pi_0 U^{\dagger} > 0$. Since $x \mapsto e^{-\beta x}$ is injective on distinct energies, π_{τ} and H_{τ} have identical spectral blocks, including degenerate eigenspaces. Thus $[\rho_{\tau}, \pi_{\tau}] = 0$ if and only if $[\rho_{\tau}, H_{\tau}] = 0$.

Second main result.— Equation (3) yields the exact optimized classification

$$C_{\tau} > 0 \iff [\rho_{\tau}, H_{\tau}] \neq 0 \iff \Sigma_{\text{M}}^{(2)} > \Sigma_{\text{M}}^{(1)}. \quad (11)$$

This supplies a binary presence criterion for the coherence contribution to isolated-drive irreversibility; its magnitude remains unquantified.

The experimental implication is one-sided unless both optimizations are certified. Let $P_{\star}$ be a certified one-copy optimizer and let M_2 be any implemented two-copy POVM. Define

$$G(M_2; P_{\star}) = \frac{1}{2} D(M_2(\rho_{\tau}^{\otimes 2}) \| M_2(\pi_{\tau}^{\otimes 2})) - D(P_{\star}(\rho_{\tau}) \| P_{\star}(\pi_{\tau})). \quad (12)$$

Then $G(M_2; P_{\star}) \leq \Sigma_{\text{M}}^{(2)} - \Sigma_{\text{M}}^{(1)}$, so a calibrated, statistically resolved value $G > 0$ implies $C_{\tau} > 0$. A null or negative restricted value is inconclusive unless M_2 is also certified globally optimal and calibration and statistical uncertainties are controlled.

For a quantitative qubit protocol, write the states in the x -readout basis as

$$\pi_f = \frac{1}{2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \quad \rho_{\tau} = \frac{1}{2} \begin{pmatrix} 1 + \delta & c L_{\delta} \\ c L_{\delta} & 1 - \delta \end{pmatrix}, \quad (13)$$

$$L_{\delta} = \frac{2\delta}{\ln[(1 + \delta)/(1 - \delta)]}.$$

Here $0 < \delta < 1$ and $c = \kappa \sqrt{1 - \delta^2}/L_{\delta}$ with $0 < \kappa < 1$, which ensures faithfulness. Prepare the displayed ρ_{τ} as the initial Gibbs state π_0 of $H_i = -\beta^{-1} \ln \rho_{\tau} + \text{const.}$ and perform an ideal sudden quench to $H_f = -\beta^{-1} \ln \pi_f + \text{const.}$ with negligible state evolution. Thus $H_0 = H_i$, $H_{\tau} = H_f$, and $\pi_{\tau} = \pi_f$ in the isolated protocol above. In the x basis, $t_{\pm} = 1 \pm \delta$ and $(\rho_{\tau})_{+-} = L_{\delta}(\pi_f)_{+-}$, so $\Omega = \text{diag}(1 + \delta, 1 - \delta)$ is stationary in the concave program (4); this certifies the x measurement as globally one-copy optimal. On two copies, apply Eq. (6) to $|+, +\rangle$ and $|-, -\rangle$, choose $\varphi = \varphi_{\star}$ to maximize Eq. (7), vary the mixing angle θ from zero, and then detect x populations. At zero angle,

$$\left. \frac{\partial D_{\text{out}}^{(2)}(\theta, \varphi_{\star})}{\partial \theta} \right|_{\theta=0} = 2\delta(1 - L_{\delta})c^2 > 0. \quad (14)$$

The two-copy rotation detailed in the Supplemental Material is a fixed two-level unitary on $\text{span}\{|+, +\rangle, |-, -\rangle\}$. It admits a fixed-size two-qubit unitary decomposition followed by local x -basis detection. The circuit statement is platform agnostic and asserts only fixed two-qubit size.

The deterministic sweep in Fig. 1 verifies the analytic slope in 80 seeded cases with worst residual below 4.3×10^{-9} and locates numerical stationary candidates of the Berta–Fawzi–Tomamichel program at 453 parameter points. On the displayed finite grid, the largest recovered activation gap is $\hat{\Delta}_{\text{num}} = 0.0214758723$ nats per copy at $\delta = 0.810$ and $\kappa = 0.99$; the explicit pair rotation attains 0.0073100901 nats per copy there and captures 24.3%–49.4% of the recovered numerical gap across the grid. At the best displayed point, the known-reference delta-method calculation [19] gives $N_5 = 38349.7$ rounds per arm for the recovered stationary candidate, rounded upward to 38350 rounds and 115050 signal copies; reference-state calibration consumes additional copies. This asymptotic count omits finite-sample bias, uncertainty in the reference law, multiple testing, drift, correlations, and systematic error. A historical backend-snapshot study for a different state ensemble and readout is preserved as non-evidentiary package provenance and is summarized briefly in the Supplemental Material. Existing experiments have measured coherence-associated entropy production, estimated coherence with two-copy collective readouts, and demonstrated two-copy collective discrimination; hypothesis-testing quantities have also been connected directly to work extraction [16, 17, 40–42]. Those experiments address other state pairs, objectives, or error models.

d. Discussion.— The theorem concerns globally optimized measurements on faithful IID pairs. Equation (11) is a binary presence criterion. The present argument yields no state-independent lower bound on $\Delta_{\star}$ in terms of C_{τ} , and the gap vanishes continuously as the pair approaches commutation. State calibration, copy correlations, drift, detector errors, and constrained gate sets can dominate an implemented witness. The classical

calculations test the formula implementation and numerical optimization. Hardware behavior, temporal stability, correlated errors, and platform-specific compilation remain outside the present scope. The argument suggests conditional extensions to measured f -divergences [27]; the relevant stationarity mean and tensor inequality are generator dependent. A general measured- f -divergence additivity classification lies outside the present result.

The novelty is narrower than “two-copy coherence measurement.” Collective work protocols and direct two-copy coherence estimation precede this work, and finite-copy collective discrimination has established Helstrom and Chernoff theories and experiments [16, 17, 43–46]. Hiai–Petz gives asymptotic recovery, Hiai–Mosonyi guarantees strict gain at some finite tensor power for non-commuting pairs, and Fang–Fawzi–Fawzi gives broader state-set superadditivity [8, 27, 28]. Hayashi and Li–Tan–Tomamichel delimit the fixed-length one-copy adaptive setting [25, 26]. Beigi–Rubboli–Tomamichel, Huang–Wilde, Fang–Fawzi–Fawzi, and Gour provide adjacent additivity, computational, variational, and axiomatic structure [13, 14, 31, 47, 48]. Ref. [18] supplies the reusable phase-adjusted projective-variation calculus. The result here adds its tensor-product measured-relative-entropy specialization: the j^n/k^n cross-score, logarithmic-mean tensor gap, explicit positive derivative, and strict gain for every $n \geq 2$ for faithful noncommuting pairs.

e. Conclusion.— For faithful finite-dimensional states, noncommutativity is exactly the obstruction to ad-

ditivity of measured relative entropy at each tensor power $n \geq 2$; strict improvement begins at two copies. For a finite-temperature thermal reference under the faithful IID assumptions, strict globally optimized two-copy gain occurs if and only if energetic coherence is nonzero. An implemented positive gain supplies a calibrated one-sided witness, while a null restricted gain leaves coherence unresolved. The explicit two-vector rotation and its direct sampling audit provide a finite-copy theoretical and numerical target within the stated classical-computation scope.

Research provenance. Maestro v0.1 produced this private Protocol 0.2 candidate through agent-assisted derivation, audit, computation, and revision. Domingos S. P. Salazar supplied the imported human source, research direction, and Program Manager stewardship. The imported human byline and source hashes are preserved in the candidate provenance record. The exact deployment snapshot of the contributing agents is unavailable and is not inferred. This disclosure records research provenance and grants no publication or release authority.

Data availability. Numerical data, local classical-simulation artifacts, and code supporting Fig. 1 are included in the local Research Object. No external repository identifier or release DOI has been assigned to this private candidate.

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