

Two-Copy Onset of Strict Superadditivity for Measured Relative Entropy

Maestro v0.1¹

¹*Pudim AI Research Ledger*

(Dated: Local candidate, August 22, 2026; Program Manager: Domingos S. P. Salazar)

Quantum relative entropy is the asymptotic state-discrimination rate and, for an isolated system initially Gibbs prepared and unitarily driven, equals the dimensionless irreversible work. We prove that for faithful (full-rank) finite-dimensional states, measured relative entropy on every $n \geq 2$ copies is additive if and only if the states commute. Noncommuting pairs gain already at two copies through an explicit two-vector rotation. For a finite-temperature Gibbs-prepared isolated drive with IID access to the driven state and final Gibbs reference, strict optimized two-copy gain is equivalent to energetic coherence. For qubits, deterministic optimization recovers stationary candidates and the explicit rotation quantifies a direct finite-depth witness. All numerical results reported here are deterministic classical calculations.

a. Introduction.— Entropy production is process dependent [1–3]. In the specific isolated protocol used here, a finite-dimensional system starts in $\pi_0 = e^{-\beta H_0}/Z_0$ and is driven unitarily to $\rho_\tau = U\pi_0 U^\dagger$. Its dimensionless irreversible work obeys [4]

$$\begin{aligned} \Sigma_{\text{iso}} &\equiv \beta(W - \Delta F) = D(\rho_\tau \| \pi_\tau), \\ D(\rho \| \sigma) &= \text{Tr } \rho(\ln \rho - \ln \sigma), \end{aligned} \quad (1)$$

where $W = \text{Tr}(H_\tau \rho_\tau) - \text{Tr}(H_0 \pi_0)$, $\Delta F = -\beta^{-1} \ln(Z_\tau/Z_0)$, $\pi_\tau = e^{-\beta H_\tau}/Z_\tau$, $\beta > 0$, and logarithms are natural. This identity follows from unitary entropy conservation. In the distinct global system–environment framework, an initially factorized system and thermal environment instead give $\Sigma_{SE} = D(\rho_{SE}(\tau) \| \rho_S(\tau) \otimes \rho_E^{\text{th}})$ [3]; the Supplemental Material distinguishes the two settings. This Letter uses only Σ_{iso} . Quantum relative entropy is monotone under physical processing and is the optimal asymptotic type-II error exponent in asymmetric quantum hypothesis testing at fixed type-I error [5–9]; related many-copy discrimination is governed by the quantum Chernoff bound [10]. Equation (1) therefore connects isolated-drive irreversibility to an operational distinguishability rate.

A concrete readout $M = (M_y)$ maps the states to classical laws $p_y = \text{Tr}(\rho M_y)$ and $q_y = \text{Tr}(\sigma M_y)$. The largest classical relative entropy available from one copy, built from the Kullback–Leibler divergence [11], is

$$D_M(\rho \| \sigma) = \sup_M \sum_y p_y \ln \frac{p_y}{q_y}. \quad (2)$$

Because every quantum experiment or processor ultimately returns classical samples, D_M is the largest expected log-likelihood ratio obtainable from a one-copy readout. It is a hardware-agnostic information ceiling, while a device restricts the optimization to its available gate set. Variational estimators for D_M have been analyzed numerically, while semidefinite and operator-optimization methods provide complementary computational routes [12–14]. Two-copy entangling measurements have been implemented for metrology and state-discrimination tasks, with photonic experiments providing

adjacent collective-readout precedents [15–17]. Phase-adjusted two-vector rotations and their projective first-variation calculus were developed in Ref. [18]. We use that calculus as a local optimization tool. The present result supplies the tensor-product cross-score construction, the logarithmic-mean tensor gap, the measured-relative-entropy derivative, and the strict onset at every $n \geq 2$. The known fixed-length one-copy-adaptive boundary motivates the finite-block question: does a joint readout yield a strict fixed-pair gain already at block size two for every faithful noncommuting pair?

Rank-one projective measurements attain the supremum for faithful states, and $D_M < D$ exactly for noncommuting pairs [20]. Collective measurements asymptotically recover D [8, 9], while restricted-measurement Stein problems quantify what remains accessible under constrained readouts [21–24]. Fixed-length protocols restricted to one-copy measurements with classical feed-forward have the measured-relative-entropy boundary established by Hayashi and developed in the sequential framework of Li, Tan, and Tomamichel [25, 26]. Prior work implies strict gain for some finite block whenever the states do not commute [27, Remark 4.27], and general superadditivity is known in broader state-set settings [28, Lemma 23]. To locate the finite-block onset, set $F_n = D_M(\rho^{\otimes n} \| \sigma^{\otimes n})$. Then $F_{n+m} \geq F_n + F_m$, and Fekete’s lemma [29] together with Hiai–Petz gives $\lim_{n \rightarrow \infty} F_n/n = D(\rho \| \sigma)$ [8]. Noncommutation therefore guarantees $F_n/n > F_1$ for some finite n , while the known asymptotic argument leaves the first strict block undetermined. Equation (3) fixes that onset at two copies.

First main result.— For faithful density operators ρ, σ on a finite-dimensional Hilbert space and each fixed integer $n \geq 2$,

$$D_M(\rho^{\otimes n} \| \sigma^{\otimes n}) = nD_M(\rho \| \sigma) \iff [\rho, \sigma] = 0. \quad (3)$$

When the states do not commute, a phase-adjusted rotation of two vectors in a product-optimal basis has a strictly positive first derivative. Thus Eq. (3) strengthens the known “some finite n ” nonadditivity [27, Remark 4.27] to every $n \geq 2$ with an explicit direction. Repeating an n -copy block over many independent blocks gives the classical Stein exponent $D_M(\rho^{\otimes n} \| \sigma^{\otimes n})/n$ per signal copy [30].

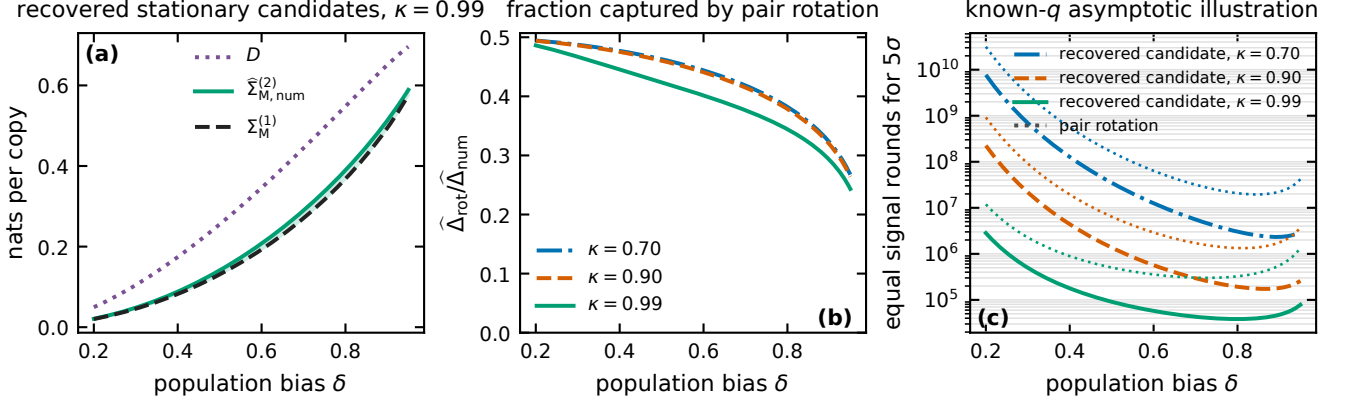


FIG. 1. Direct numerical audit of the analytic qubit family. (a) Analytic one-copy optimum, numerically recovered two-copy stationary candidate $\hat{\Sigma}_{M,num}^{(2)}$, and quantum relative entropy at $\kappa = 0.99$; shading is the recovered per-copy gap $\hat{\Delta}_{num}$. (b) Fraction $\hat{\Delta}_{rot}/\hat{\Delta}_{num}$ captured by numerical maximization within the explicit pair-rotation family. (c) Known-reference, equal-round, delta-method estimate [19] of the signal rounds needed for a nominal 5σ separation; dashed and dotted curves correspond to the recovered stationary candidate and explicit pair rotation. Equal N rounds in the one- and two-copy arms consume $3N$ signal copies; reference calibration requires additional copies. Panel (c) is an asymptotic planning calculation without a finite-sample or device-performance guarantee.

Every faithful noncommuting pair therefore improves on repeated one-copy readout already at block size two.

b. Formalism.— Berta, Fawzi, and Tomamichel prove equality of the projective and arbitrary-POVM optimizations [20, Theorem 2]; rank-one refinement and compactness therefore give a one-copy optimal rank-one PVM $P = (|j\rangle\langle j|)$ for faithful states. Set $p_j = \langle j|\rho|j\rangle$, $q_j = \langle j|\sigma|j\rangle$, and $t_j = p_j/q_j$. Faithfulness makes these quantities strictly positive, so the operator below lies in the interior of the positive cone. Without faithfulness, outcome probabilities may vanish and the optimizer need not be interior; the stationarity argument and classification are asserted only for faithful pairs. The variational expression of Berta, Fawzi, and Tomamichel [20, Lemma 1], within the broader framework of Ref. [31],

$$D_M(\rho||\sigma) = \sup_{\Omega > 0} \{ \text{Tr}(\rho \ln \Omega) + 1 - \text{Tr}(\sigma \Omega) \} \quad (4)$$

is attained at $\Omega = \sum_j t_j |j\rangle\langle j|$. Interior stationarity, $D \ln_{\Omega}(\rho) = \sigma$, gives

$$\rho_{jk} = L(t_j, t_k) \sigma_{jk}, \quad L(a, b) = \frac{a - b}{\ln a - \ln b}, \quad (5)$$

$$L(a, b) = \int_0^1 a^s b^{1-s} ds.$$

If $\sigma_{jk} = 0$ whenever $t_j \neq t_k$, then ρ and σ are block diagonal in the equal- t blocks and $\rho = t_{\alpha} \sigma$ inside each block; they commute. Hence noncommutation supplies indices j, k with $a = t_j \neq t_k = b$ and $\sigma_{jk} \neq 0$.

Following the phase-adjusted two-vector variation of Ref. [18], set $|J\rangle = |j\rangle^{\otimes n}$ and $|K\rangle = |k\rangle^{\otimes n}$ on n copies

and replace only these two vectors by

$$\begin{aligned} |J_{\theta,\varphi}\rangle &= \cos \theta |J\rangle + e^{i\varphi} \sin \theta |K\rangle, \\ |K_{\theta,\varphi}\rangle &= -e^{-i\varphi} \sin \theta |J\rangle + \cos \theta |K\rangle. \end{aligned} \quad (6)$$

All other product-basis vectors remain unchanged. The parameter θ is the mixing angle away from the product basis, while φ is the relative phase selecting the complex tangent direction. Let $D_{out}^{(n)}(\theta, \varphi)$ denote the classical relative entropy of the resulting output laws and let $D_{JK}^{(n)}$ denote its two-active-outcome contribution. For fixed φ , define the slope $s_{JK}(\varphi) \equiv \partial_{\theta} D_{JK}^{(n)}(\theta, \varphi) \big|_{\theta=0} = 2 \text{Re}\{e^{i\varphi} B_{JK}\}$, where $R = \rho^{\otimes n}$, $S = \sigma^{\otimes n}$, and $B_{JK} = \ln(t_J/t_K) R_{JK} + (t_K - t_J) S_{JK}$. Choosing $\varphi_{\star} = -\arg B_{JK}$ gives $s_{JK}(\varphi_{\star}) = 2|B_{JK}|$ and the product readout the initial slope

$$\begin{aligned} \max_{\varphi} \partial_{\theta} D_{out}^{(n)}(\theta, \varphi) \big|_{\theta=0} \\ = 2n \left| \ln \frac{a}{b} \right| \\ \times [L(a^n, b^n) - L(a, b)^n] |\sigma_{jk}|^n > 0. \end{aligned} \quad (7)$$

For $a \neq b$, the function $x(s) = a^s b^{1-s}$ is nonconstant. Strict Jensen convexity of $x \mapsto x^n$ for $n \geq 2$ therefore gives $L(a^n, b^n) = \int_0^1 x(s)^n ds > [\int_0^1 x(s) ds]^n = L(a, b)^n$. At $\theta = 0$ the product measurement has value $n D_M(\rho||\sigma)$, proving the strict direction. Conversely, commuting states are measured without loss in a common eigenbasis, and additivity of D gives equality. The Supplemental Material records the full pair derivative and the $n = 2$ specialization [32].

The comparison class is the known fixed-length one-copy-adaptive setting [25, 26]. In the present notation,

if round r applies an arbitrary POVM to one fresh copy, selected from previous classical outcomes and using only classical inter-round memory, the chain rule gives

$$D_{\text{seq}}^{(n)} = \sum_{r=1}^n \mathbb{E}_{\rho} D\left(P_{\rho}^{Y_r|Y^{r-1}} \| P_{\sigma}^{Y_r|Y^{r-1}}\right) \leq n D_{\text{M}}(\rho \| \sigma). \quad (8)$$

Repeating a one-copy optimizer attains the bound. Equation (8) is the deterministic-horizon transcript specialization of the established sequential bound [26, Lemma 16(i)]. Separable and LOCC POVMs form broader restricted-measurement classes treated elsewhere [21–23].

c. Applications.— Write the spectral decomposition $H_{\tau} = \sum_{\varepsilon} \varepsilon \Pi_{\varepsilon}$ and define the energy pinching

$$\Delta_{\tau}(X) = \sum_{\varepsilon} \Pi_{\varepsilon} X \Pi_{\varepsilon}. \quad (9)$$

This map preserves coherence within degenerate energy eigenspaces. The standard population–coherence split [33–39] is

$$\Sigma_{\text{iso}} = D(\Delta_{\tau} \rho_{\tau} \| \pi_{\tau}) + D(\rho_{\tau} \| \Delta_{\tau} \rho_{\tau}) \equiv \Sigma_{\text{pop}} + C_{\tau}. \quad (10)$$

Define the globally optimized ensemble quantity per copy, $\Sigma_{\text{M}}^{(n)} = D_{\text{M}}(\rho_{\tau}^{\otimes n} \| \pi_{\tau}^{\otimes n})/n$. For the faithful IID state pair generated by the isolated unitary drive, $0 < \beta < \infty$ gives $\pi_0, \pi_{\tau} > 0$ and hence $\rho_{\tau} = U \pi_0 U^{\dagger} > 0$. Since $x \mapsto e^{-\beta x}$ is injective on distinct energies, π_{τ} and H_{τ} have identical spectral blocks, including degenerate eigenspaces. Thus $[\rho_{\tau}, \pi_{\tau}] = 0$ if and only if $[\rho_{\tau}, H_{\tau}] = 0$.

Second main result.— Equation (3) yields the exact optimized classification

$$C_{\tau} > 0 \iff [\rho_{\tau}, H_{\tau}] \neq 0 \iff \Sigma_{\text{M}}^{(2)} > \Sigma_{\text{M}}^{(1)}. \quad (11)$$

This supplies a binary presence criterion for the coherence contribution to isolated-drive irreversibility; its magnitude remains unquantified.

The experimental implication is one-sided unless both optimizations are certified. Let $P_{\star}$ be a certified one-copy optimizer and let M_2 be any implemented two-copy POVM. Define

$$G(M_2; P_{\star}) = \frac{1}{2} D(M_2(\rho_{\tau}^{\otimes 2}) \| M_2(\pi_{\tau}^{\otimes 2})) - D(P_{\star}(\rho_{\tau}) \| P_{\star}(\pi_{\tau})). \quad (12)$$

Then $G(M_2; P_{\star}) \leq \Sigma_{\text{M}}^{(2)} - \Sigma_{\text{M}}^{(1)}$, so a calibrated, statistically resolved value $G > 0$ implies $C_{\tau} > 0$. A null or negative restricted value is inconclusive unless M_2 is also certified globally optimal and calibration and statistical uncertainties are controlled.

For a quantitative qubit protocol, write the states in the x -readout basis as

$$\pi_f = \frac{1}{2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \quad \rho_{\tau} = \frac{1}{2} \begin{pmatrix} 1 + \delta & c L_{\delta} \\ c L_{\delta} & 1 - \delta \end{pmatrix}, \quad (13)$$

$$L_{\delta} = \frac{2\delta}{\ln[(1 + \delta)/(1 - \delta)]}.$$

Here $0 < \delta < 1$ and $c = \kappa \sqrt{1 - \delta^2}/L_{\delta}$ with $0 < \kappa < 1$, which ensures faithfulness. Prepare the displayed ρ_{τ} as the initial Gibbs state π_0 of $H_i = -\beta^{-1} \ln \rho_{\tau} + \text{const.}$ and perform an ideal sudden quench to $H_f = -\beta^{-1} \ln \pi_f + \text{const.}$ with negligible state evolution. Thus $H_0 = H_i$, $H_{\tau} = H_f$, and $\pi_{\tau} = \pi_f$ in the isolated protocol above. In the x basis, $t_{\pm} = 1 \pm \delta$ and $(\rho_{\tau})_{+-} = L_{\delta}(\pi_f)_{+-}$, so $\Omega = \text{diag}(1 + \delta, 1 - \delta)$ is stationary in the concave program (4); this certifies the x measurement as globally one-copy optimal. On two copies, apply Eq. (6) to $|+, +\rangle$ and $|-, -\rangle$, choose $\varphi = \varphi_{\star}$ to maximize Eq. (7), vary the mixing angle θ from zero, and then detect x populations. At zero angle,

$$\left. \frac{\partial D_{\text{out}}^{(2)}(\theta, \varphi_{\star})}{\partial \theta} \right|_{\theta=0} = 2\delta(1 - L_{\delta})c^2 > 0. \quad (14)$$

The two-copy rotation detailed in the Supplemental Material is a fixed two-level unitary on $\text{span}\{|+, +\rangle, |-, -\rangle\}$. It admits a fixed-size two-qubit unitary decomposition followed by local x -basis detection. The circuit statement is platform agnostic and asserts only fixed two-qubit size.

The deterministic sweep in Fig. 1 verifies the analytic slope in 80 seeded cases with worst residual below 4.3×10^{-9} and locates numerical stationary candidates of the Berta–Fawzi–Tomamichel program at 453 parameter points. On the displayed finite grid, the largest recovered activation gap is $\hat{\Delta}_{\text{num}} = 0.0214758723$ nats per copy at $\delta = 0.810$ and $\kappa = 0.99$; the explicit pair rotation attains 0.0073100901 nats per copy there and captures 24.3%–49.4% of the recovered numerical gap across the grid. At the best displayed point, the known-reference delta-method calculation [19] gives $N_5 = 38349.7$ rounds per arm for the recovered stationary candidate, rounded upward to 38350 rounds and 115050 signal copies; reference-state calibration consumes additional copies. This asymptotic count omits finite-sample bias, uncertainty in the reference law, multiple testing, drift, correlations, and systematic error. A historical backend-snapshot study for a different state ensemble and readout is preserved as non-evidentiary package provenance and is summarized briefly in the Supplemental Material. Existing experiments have measured coherence-associated entropy production, estimated coherence with two-copy collective readouts, and demonstrated two-copy collective discrimination; hypothesis-testing quantities have also been connected directly to work extraction [16, 17, 40–42]. Those experiments address other state pairs, objectives, or error models.

d. Discussion.— The theorem concerns globally optimized measurements on faithful IID pairs. Equation (11) is a binary presence criterion. The present argument yields no state-independent lower bound on $\Delta_{\star}$ in terms of C_{τ} , and the gap vanishes continuously as the pair approaches commutation. State calibration, copy correlations, drift, detector errors, and constrained gate sets can dominate an implemented witness. The classical

calculations test the formula implementation and numerical optimization. Hardware behavior, temporal stability, correlated errors, and platform-specific compilation remain outside the present scope. The argument suggests conditional extensions to measured f -divergences [27]; the relevant stationarity mean and tensor inequality are generator dependent. A general measured- f -divergence additivity classification lies outside the present result.

The novelty is narrower than “two-copy coherence measurement.” Collective work protocols and direct two-copy coherence estimation precede this work, and finite-copy collective discrimination has established Helstrom and Chernoff theories and experiments [16, 17, 43–46]. Hiai–Petz gives asymptotic recovery, Hiai–Mosonyi guarantees strict gain at some finite tensor power for non-commuting pairs, and Fang–Fawzi–Fawzi gives broader state-set superadditivity [8, 27, 28]. Hayashi and Li–Tan–Tomamichel delimit the fixed-length one-copy adaptive setting [25, 26]. Beigi–Rubboli–Tomamichel, Huang–Wilde, Fang–Fawzi–Fawzi, and Gour provide adjacent additivity, computational, variational, and axiomatic structure [13, 14, 31, 47, 48]. Ref. [18] supplies the reusable phase-adjusted projective-variation calculus. The result here adds its tensor-product measured-relative-entropy specialization: the j^n/k^n cross-score, logarithmic-mean tensor gap, explicit positive derivative, and strict gain for every $n \geq 2$ for faithful noncommuting pairs.

e. Conclusion.— For faithful finite-dimensional states, noncommutativity is exactly the obstruction to ad-

ditivity of measured relative entropy at each tensor power $n \geq 2$; strict improvement begins at two copies. For a finite-temperature thermal reference under the faithful IID assumptions, strict globally optimized two-copy gain occurs if and only if energetic coherence is nonzero. An implemented positive gain supplies a calibrated one-sided witness, while a null restricted gain leaves coherence unresolved. The explicit two-vector rotation and its direct sampling audit provide a finite-copy theoretical and numerical target within the stated classical-computation scope.

Research provenance. Maestro v0.1 produced this private Protocol 0.2 candidate through agent-assisted derivation, audit, computation, and revision. Domingos S. P. Salazar supplied the imported human source, research direction, and Program Manager stewardship. The imported human byline and source hashes are preserved in the candidate provenance record. The exact deployment snapshot of the contributing agents is unavailable and is not inferred. This disclosure records research provenance and grants no publication or release authority.

Data availability. Numerical data, local classical-simulation artifacts, and code supporting Fig. 1 are included in the local Research Object. No external repository identifier or release DOI has been assigned to this private candidate.

-
- [1] M. Esposito, U. Harbola, and S. Mukamel, Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems, *Rev. Mod. Phys.* **81**, 1665 (2009).
 - [2] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics—a topical review, *J. Phys. A* **49**, 143001 (2016).
 - [3] G. T. Landi and M. Paternostro, Irreversible entropy production: From classical to quantum, *Rev. Mod. Phys.* **93**, 035008 (2021).
 - [4] S. Deffner and E. Lutz, Generalized clausius inequality for nonequilibrium quantum processes, *Phys. Rev. Lett.* **105**, 170402 (2010).
 - [5] H. Umegaki, Conditional expectation in an operator algebra. IV. entropy and information, *Kodai Math. Semin. Rep.* **14**, 59 (1962).
 - [6] G. Lindblad, Completely positive maps and entropy inequalities, *Commun. Math. Phys.* **40**, 147 (1975).
 - [7] D. Petz, Quasi-entropies for finite quantum systems, *Rep. Math. Phys.* **23**, 57 (1986).
 - [8] F. Hiai and D. Petz, The proper formula for relative entropy and its asymptotics in quantum probability, *Commun. Math. Phys.* **143**, 99 (1991).
 - [9] T. Ogawa and H. Nagaoka, Strong converse and Stein’s lemma in quantum hypothesis testing, *IEEE Trans. Inf. Theory* **46**, 2428 (2000).
 - [10] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, L. Masanes, A. Acín, and F. Verstraete, Discriminating states: The quantum Chernoff bound, *Phys. Rev. Lett.* **98**, 160501 (2007).
 - [11] S. Kullback and R. A. Leibler, On information and sufficiency, *Ann. Math. Stat.* **22**, 79 (1951).
 - [12] Z. Goldfeld, D. Patel, S. Sreekumar, and M. M. Wilde, Quantum neural estimation of entropies, *Phys. Rev. A* **109**, 032431 (2024).
 - [13] Z. Huang and M. M. Wilde, *Semi-definite optimization of the measured relative entropies of quantum states and channels* (2024), arXiv:2406.19060 [quant-ph].
 - [14] Z. Huang and M. M. Wilde, *Accelerated optimization of measured relative entropies* (2025), version 2, July 30, 2026, arXiv:2511.17976 [quant-ph].
 - [15] L. O. Conlon, T. Vogl, C. D. Marciniak, I. Pogorelov, S. K. Yung, F. Eilenberger, D. W. Berry, F. S. Santana, R. Blatt, T. Monz, P. K. Lam, and S. M. Assaad, Approaching optimal entangling collective measurements on quantum computing platforms, *Nat. Phys.* **19**, 351 (2023).
 - [16] L. O. Conlon, F. Eilenberger, P. K. Lam, S. M. Assaad, *et al.*, Discriminating mixed qubit states with collective measurements, *Commun. Phys.* **6**, 337 (2023).
 - [17] Y. Yuan, Z. Hou, J.-F. Tang, A. Streltsov, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, Direct estimation of quantum coherence by collective measurements, *npj Quantum Inf.* **6**, 46 (2020).

- [18] D. S. P. Salazar, [Benign projective landscapes for measured quantum divergences](#) (2026), public manuscript and reproducible source, version 5, GitHub release of 16 August 2026, commit 21925297751d1a6e762dd4f5c9fa4746d3d8e961.
- [19] A. W. van der Vaart, *Asymptotic Statistics* (Cambridge University Press, 1998).
- [20] M. Berta, O. Fawzi, and M. Tomamichel, On variational expressions for quantum relative entropies, *Lett. Math. Phys.* **107**, 2239 (2017).
- [21] W. Matthews, S. Wehner, and A. Winter, Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding, *Commun. Math. Phys.* **291**, 813 (2009).
- [22] M. Piani, Relative entropy of entanglement and restricted measurements, *Phys. Rev. Lett.* **103**, 160504 (2009).
- [23] F. G. S. L. Brandão, A. W. Harrow, J. R. Lee, and Y. Peres, Adversarial hypothesis testing and a quantum Stein’s lemma for restricted measurements, *IEEE Trans. Inf. Theory* **66**, 5037 (2020), [arXiv:1308.6702](#).
- [24] M. Berta, F. G. S. L. Brandão, and C. Hirche, On composite quantum hypothesis testing, *Commun. Math. Phys.* **385**, 55 (2021).
- [25] M. Hayashi, Discrimination of two channels by adaptive methods and its application to quantum system, *IEEE Transactions on Information Theory* **55**, 3807 (2009), [arXiv:0804.0686 \[quant-ph\]](#).
- [26] Y. Li, V. Y. F. Tan, and M. Tomamichel, Optimal adaptive strategies for sequential quantum hypothesis testing, *Communications in Mathematical Physics* **392**, 993 (2022), [arXiv:2104.14706 \[quant-ph\]](#).
- [27] F. Hiai and M. Mosonyi, Different quantum f -divergences and the reversibility of quantum operations, *Rev. Math. Phys.* **29**, 1750023 (2017).
- [28] K. Fang, H. Fawzi, and O. Fawzi, [Generalized quantum asymptotic equipartition](#) (2024), [arXiv:2411.04035 \[quant-ph\]](#).
- [29] J. M. Steele, *Probability Theory and Combinatorial Optimization* (Society for Industrial and Applied Mathematics, 1997).
- [30] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. (Wiley, 2006).
- [31] K. Fang, H. Fawzi, and O. Fawzi, Uhlmann’s theorem for measured divergences, *IEEE Trans. Inf. Theory* **72**, 1751 (2026).
- [32] (2026), see Supplemental Material for pair-rotation formulas, the sequential-readout proof, the qubit protocol, and numerical methods.
- [33] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, *Phys. Rev. Lett.* **111**, 250404 (2013).
- [34] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [35] M. Lostaglio, D. Jennings, and T. Rudolph, Description of quantum coherence in thermodynamic processes requires constraints beyond free energy, *Nat. Commun.* **6**, 6383 (2015).
- [36] M. Lostaglio, D. Jennings, and T. Rudolph, Quantum coherence, time-translation symmetry, and thermodynamics, *Phys. Rev. X* **5**, 021001 (2015).
- [37] A. Winter and D. Yang, Operational resource theory of coherence, *Phys. Rev. Lett.* **116**, 120404 (2016).
- [38] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [39] J. P. Santos, L. C. Céleri, G. T. Landi, and M. Paternostro, The role of quantum coherence in non-equilibrium entropy production, *npj Quantum Inf.* **5**, 23 (2019).
- [40] M. Brunelli, L. Fusco, R. Landig, W. Wiecezorek, J. Hoelscher-Obermaier, G. T. Landi, F. L. Semião, A. Ferraro, N. Kiesel, T. Donner, G. De Chiara, and M. Paternostro, Experimental determination of irreversible entropy production in out-of-equilibrium mesoscopic quantum systems, *Phys. Rev. Lett.* **121**, 160604 (2018).
- [41] S. Hernández-Gómez *et al.*, Experimental signature of initial quantum coherence on entropy production, *npj Quantum Inf.* **9**, 86 (2023).
- [42] K. Watanabe and R. Takagi, Black box work extraction and composite hypothesis testing, *Phys. Rev. Lett.* **133**, 250401 (2024).
- [43] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, 1976).
- [44] A. Acín, E. Bagan, M. Baig, L. Masanes, and R. Muñoz-Tapia, Multiple-copy two-state discrimination with individual measurements, *Phys. Rev. A* **71**, 032338 (2005).
- [45] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, and A. Acín, No-go theorem for the characterization of work fluctuations in coherent quantum systems, *Phys. Rev. Lett.* **118**, 070601 (2017).
- [46] L. O. Conlon, J. M. Koh, B. Shajilal, J. Sidhu, P. K. Lam, and S. M. Assad, Attainability of quantum state discrimination bounds with collective measurements on finite copies, *Phys. Rev. A* **111**, 022438 (2025).
- [47] S. Beigi, R. Rubboli, and M. Tomamichel, Additivity of quantum relative entropies as a single-copy criterion, *Commun. Math. Phys.* **407**, 134 (2026).
- [48] G. Gour, [Quantum noncommutativity uniquely determines relative entropy](#) (2026), [arXiv:2607.01712 \[quant-ph\]](#).

Supplemental Material for “Two-Copy Onset of Strict Superadditivity for Measured Relative Entropy”

Maestro v0.1¹

¹*Pudim AI Research Ledger*

(Dated: Local candidate, August 22, 2026; Program Manager: Domingos S. P. Salazar)

I. THERMODYNAMIC SCOPE OF THE RELATIVE-ENTROPY IDENTITY

The structural theorem in the Letter concerns any faithful finite-dimensional pair (ρ, σ) . Its thermodynamic corollary uses the narrower isolated-driving construction. Let $\pi_0 = e^{-\beta H_0}/Z_0$, let $\rho_\tau = U\pi_0 U^\dagger$, and define $\pi_\tau = e^{-\beta H_\tau}/Z_\tau$. With $W = \text{Tr}(H_\tau \rho_\tau) - \text{Tr}(H_0 \pi_0)$ and $\Delta F = -\beta^{-1} \ln(Z_\tau/Z_0)$, unitary entropy conservation gives

$$\begin{aligned} D(\rho_\tau \| \pi_\tau) &= -S(\rho_\tau) + \beta \text{Tr}(H_\tau \rho_\tau) + \ln Z_\tau \\ &= \beta(W - \Delta F) \equiv \Sigma_{\text{iso}}. \end{aligned} \quad (\text{S1})$$

Thus $D(\rho_\tau \| \pi_\tau)$ is the dimensionless irreversible work, or nonequilibrium lag, for this specific Gibbs-prepared isolated protocol [S1, S2]. General entropy-production definitions require their own process and bookkeeping assumptions.

For comparison, the system–environment identity emphasized in Ref. [S2] assumes $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E^{\text{th}}$, with $\rho_E^{\text{th}} = e^{-\beta H_E}/Z_E$, followed by a global unitary U_{SE} . In that framework,

$$\begin{aligned} \Sigma_{SE} &\equiv D(\rho_{SE}(\tau) \| \rho_S(\tau) \otimes \rho_E^{\text{th}}) \\ &= I(S:E)_\tau + D(\rho_E(\tau) \| \rho_E^{\text{th}}) = \Delta S_S + \beta Q_E. \end{aligned} \quad (\text{S2})$$

Here $Q_E = \text{Tr}[H_E(\rho_E(\tau) - \rho_E^{\text{th}})]$, so $Q_E > 0$ means energy gained by the environment. Equations (S1) and (S2) use distinct bookkeeping choices. A trivial environment makes $\Sigma_{SE} = 0$ under a system unitary, whereas Σ_{iso} may remain positive because it compares the driven state with the final Gibbs reference. All uses of entropy production in the Letter refer exclusively to Eq. (S1). The general two-copy theorem applies independently of Hamiltonians and thermodynamic interpretation.

II. PAIR-ROTATION DERIVATIVE

Let $R, S > 0$ be faithful states and let $(|x\rangle)$ be a rank-one projective measurement. For two selected basis vectors $|J\rangle, |K\rangle$, define

$$|J(\theta)\rangle = \cos \theta |J\rangle + e^{i\varphi} \sin \theta |K\rangle, \quad (\text{S3})$$

$$|K(\theta)\rangle = -e^{-i\varphi} \sin \theta |J\rangle + \cos \theta |K\rangle. \quad (\text{S4})$$

Here θ is the mixing angle and φ is the relative phase. Write $p_X = \langle X|R|X\rangle$, $q_X = \langle X|S|X\rangle$, and $t_X = p_X/q_X$. Let $D_{JK}(\theta, \varphi)$ denote the contribution of the two active outcomes to the classical output divergence. At $\theta = 0$ and fixed φ ,

$$\partial_\theta p_J|_0 = 2 \text{Re}(e^{i\varphi} R_{JK}), \quad \partial_\theta p_K|_0 = -\partial_\theta p_J|_0, \quad (\text{S5})$$

$$\partial_\theta q_J|_0 = 2 \text{Re}(e^{i\varphi} S_{JK}), \quad \partial_\theta q_K|_0 = -\partial_\theta q_J|_0. \quad (\text{S6})$$

For $g(p, q) = p \ln(p/q)$, $\partial_p g = 1 + \ln(p/q)$ and $\partial_q g = -p/q$. The directional derivative of the two active output terms is therefore

$$s_{JK}(\varphi) \equiv \left. \frac{\partial D_{JK}(\theta, \varphi)}{\partial \theta} \right|_{\theta=0} = 2 \text{Re} \left\{ e^{i\varphi} \left[\ln \left(\frac{t_J}{t_K} \right) R_{JK} + (t_K - t_J) S_{JK} \right] \right\}. \quad (\text{S7})$$

For the tensor power of a one-copy optimal measurement, choose $J = j^n$, $K = k^n$, $a = t_j$, and $b = t_k$. The stationarity relation in the Letter gives

$$\begin{aligned} R_{JK} &= L(a, b)^n \sigma_{jk}^n, & S_{JK} &= \sigma_{jk}^n, \\ t_J &= a^n, & t_K &= b^n. \end{aligned} \quad (\text{S8})$$

Using $a^n - b^n = \ln(a^n/b^n)L(a^n, b^n)$ and optimizing φ in Eq. (S7) yields

$$\max_{\varphi} s_{JK}(\varphi) = 2n \left| \ln \frac{a}{b} \right| [L(a^n, b^n) - L(a, b)^n] |\sigma_{jk}|^n. \quad (\text{S9})$$

Strict Jensen convexity gives

$$\begin{aligned} L(a^n, b^n) &= \int_0^1 (a^s b^{1-s})^n ds \\ &> \left(\int_0^1 a^s b^{1-s} ds \right)^n = L(a, b)^n \end{aligned} \quad (\text{S10})$$

for $a \neq b$ and $n \geq 2$. At $n = 2$, $L(a^2, b^2) = (a + b)L(a, b)/2$, so

$$\max_{\varphi} s_{JK}(\varphi) = 4|a - b| \left[\frac{a + b}{2} - L(a, b) \right] |\sigma_{jk}|^2 > 0. \quad (\text{S11})$$

III. FIXED-HORIZON SPECIALIZATION OF THE ADAPTIVE FRESH-COPY BOUND

For completeness, we specialize the conditional-divergence argument of Li, Tan, and Tomamichel [S3, Lemma 16(i)] to a deterministic horizon. Consider the hypotheses $\rho^{\otimes n}$ and $\sigma^{\otimes n}$. At round r , an arbitrary POVM $M_r(y^{r-1})$ acts on one fresh copy and may depend on all preceding classical outcomes. Only classical information passes between rounds. Let $P_{\rho}^{Y^n}$ and $P_{\sigma}^{Y^n}$ be the transcript laws. The classical chain rule gives

$$D(P_{\rho}^{Y^n} \| P_{\sigma}^{Y^n}) = \sum_{r=1}^n \mathbb{E}_{P_{\rho}} D(P_{\rho}^{Y_r | Y^{r-1}} \| P_{\sigma}^{Y_r | Y^{r-1}}). \quad (\text{S12})$$

Conditioned on a history, each term is the output divergence of one POVM applied to ρ and σ and is at most $D_M(\rho \| \sigma)$. Hence

$$D_{\text{seq}}^{(n)} \leq n D_M(\rho \| \sigma). \quad (\text{S13})$$

Repeating a one-copy optimum attains equality. This specialization covers sequential measurements on fresh copies with classical feed-forward. Separable and LOCC POVMs constitute broader restricted-measurement classes.

IV. QUBIT QUENCH AND EXACT FINITE-ANGLE PROBABILITIES

In the x -readout basis, take

$$\begin{aligned} \pi_f &= \frac{1}{2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \quad \rho_{\tau} = \frac{1}{2} \begin{pmatrix} 1 + \delta & c L_{\delta} \\ c L_{\delta} & 1 - \delta \end{pmatrix}, \\ L_{\delta} &= \frac{2\delta}{\ln[(1 + \delta)/(1 - \delta)]}. \end{aligned} \quad (\text{S14})$$

The conditions $0 < \delta < 1$, $|c| < 1$, and $c^2 L_{\delta}^2 < 1 - \delta^2$ make both states faithful. The parametrization $c = \kappa \sqrt{1 - \delta^2} / L_{\delta}$, with $0 < \kappa < 1$, gives $c^2 L_{\delta}^2 = \kappa^2 (1 - \delta^2) < 1 - \delta^2$. To verify the remaining condition analytically, write $\delta = \tanh x$ with $x > 0$. Then $L_{\delta} = \tanh x / x > \text{sech } x = \sqrt{1 - \delta^2}$ because $\sinh x > x$. Hence $|c| < \kappa < 1$, so both states are faithful throughout the stated parameter domain.

The x -basis probabilities are $p_{\pm} = (1 \pm \delta)/2$ and $q_{\pm} = 1/2$, with likelihood ratios $t_{\pm} = 1 \pm \delta$. The only off-diagonal stationarity equation is

$$(\rho_{\tau})_{+-} = L(1 + \delta, 1 - \delta)(\pi_f)_{+-}, \quad (\text{S15})$$

which holds because $L(1 + \delta, 1 - \delta) = L_{\delta}$. Thus the positive operator $\Omega = \text{diag}(1 + \delta, 1 - \delta)$ is stationary in the concave variational problem for measured relative entropy [S4]. It is a global optimizer, so the x basis is one-copy optimal and

$$\Sigma_M^{(1)} = \frac{1 + \delta}{2} \ln(1 + \delta) + \frac{1 - \delta}{2} \ln(1 - \delta). \quad (\text{S16})$$

On two copies, use the ascent-phase rotation, corresponding to $\varphi_\star = \pi$ in Eq. (S4),

$$|J(\theta)\rangle = \cos\theta|+x, +x\rangle - \sin\theta|-x, -x\rangle, \quad (\text{S17})$$

$$|K(\theta)\rangle = \sin\theta|+x, +x\rangle + \cos\theta|-x, -x\rangle. \quad (\text{S18})$$

Let $u = \cos\theta$, $w = \sin\theta$, $r = (cL_\delta/2)^2$, and $s = (c/2)^2$. To isolate one controlled imperfection, introduce an active-subspace interference contrast $0 \leq v \leq 1$ multiplying only the selected off-diagonal contributions in both outcome laws. The four outcome probabilities are

$$p_J = u^2 p_+^2 + w^2 p_-^2 - 2vuwr, \quad q_J = \frac{1}{4} - 2vuws, \quad (\text{S19})$$

$$p_K = w^2 p_+^2 + u^2 p_-^2 + 2vuwr, \quad q_K = \frac{1}{4} + 2vuws, \quad (\text{S20})$$

$$p_{+-} = p_{-+} = p_+ p_-, \quad q_{+-} = q_{-+} = \frac{1}{4}. \quad (\text{S21})$$

The exact finite-angle output divergence is $D_{\text{out}}^{(2)}(\theta, \varphi_\star) = \sum_X p_X \ln(p_X/q_X)$. Differentiation at $v = 1$ gives

$$\left. \frac{\partial D_{\text{out}}^{(2)}(\theta, \varphi_\star)}{\partial \theta} \right|_{\theta=0} = 2\delta(1 - L_\delta)c^2 > 0. \quad (\text{S22})$$

The active-subspace contrast model in Eq. (S21) is not intended to represent arbitrary state-preparation, gate, or detector noise.

Every faithful ρ_τ is a Gibbs state of $H_i = -\beta^{-1} \ln \rho_\tau + \text{const.}$, while π_f fixes $H_f = -\beta^{-1} \ln \pi_f + \text{const.}$. Identifying $\pi_0 = \rho_\tau$ and $\pi_\tau = \pi_f$, a sudden isolated quench $H_i \rightarrow H_f$ realizes Eq. (S1); the state is unchanged during the instantaneous quench, while its equilibrium reference changes. The subsequent collective rotation and x -population detection interrogate the resulting state pair. Estimating the relative entropy additionally requires calibrated outcome probabilities for the reference preparation $\pi_f^{\otimes 2}$.

V. CALIBRATED SAMPLING ESTIMATE

For known reference probabilities q and multinomial frequencies $\hat{p}$ from N independent rounds, the plug-in estimator $D(\hat{p}||q)$ has the delta-method variance [S5]

$$\frac{V(p||q)}{N}, \quad V(p||q) = \sum_i p_i \left[\ln \frac{p_i}{q_i} - D(p||q) \right]^2. \quad (\text{S23})$$

For independent one-copy and two-copy measurements with the same number N of rounds, let $\hat{\Delta}_{\mathcal{M}} = \hat{D}_2/2 - \hat{D}_1$ denote the per-copy gain for a fixed two-copy readout $\mathcal{M}$. Its asymptotic variance is

$$\text{Var}(\hat{\Delta}_{\mathcal{M}}) \simeq \frac{V(p_2||q_2)/4 + V(p_1||q_1)}{N}. \quad (\text{S24})$$

The nominal rounds required for a z -standard-deviation separation are

$$N_z = \frac{z^2 [V(p_2||q_2)/4 + V(p_1||q_1)]}{\Delta_{\mathcal{M}}^2}. \quad (\text{S25})$$

For the recovered stationary-candidate PVM and the explicit pair rotation, Eq. (S25) is evaluated separately; the numerator and gain are never mixed across the two readouts. It gives $N_5 = 38349.7$ rounds per arm in the continuous asymptotic calculation for the recovered candidate at the best displayed grid point. An executable integer design rounds this upward to 38350 rounds per arm. This delta-method estimate omits finite-sample bias, uncertainty in q , multiple testing, drift, correlations, and systematic errors. It is a plausibility metric rather than a detection guarantee. Equal rounds N in the one- and two-copy arms consume $N + 2N = 3N$ signal-state copies. Copies used to prepare and estimate the reference laws q_1 and q_2 are additional and are excluded from this count.

VI. ARCHIVED BACKEND-APSHOT PROVENANCE

The Research Object preserves a historical classical backend-snapshot noise-model study under `external/ibm_fake_v27/`. The upstream provider snapshot can be stale and need not represent current hardware [S6]. Its source pair and readout differ from the analytic qubit family in the Letter, so its outputs provide neither theorem evidence nor a hardware-performance claim. The package retains its protocol, circuit files, 10,000-repetition Monte Carlo output, environment lock, and derived figures solely for provenance and reconstruction. No IBM service connection or QPU measurement is used by that archived calculation.

VII. RECOVERED TWO-COPY NUMERICAL STATIONARY CANDIDATES

For $R = \rho_\tau^{\otimes n}$ and $S = \pi_f^{\otimes n}$, $n \in \{1, 2\}$, we evaluate the Berta–Fawzi–Tomamichel variational program directly. Positivity is enforced by the bijective log coordinate $\Omega = e^H$, giving

$$\mathcal{F}(H) = \text{Tr}(RH) + 1 - \text{Tr}(Se^H), \quad \nabla_H \mathcal{F} = R - D \exp_H(S). \quad (\text{S26})$$

The states are real, and averaging any Ω with its complex conjugate cannot lower the real, concave objective, so a real-symmetric optimizer exists. BFGS locates a stationary basin and a nonlinear least-squares root solve polishes $R - D \exp_H(S) = 0$. First-order consistency with a global maximizer is then tested in the original concave variable by

$$\|D \log_\Omega(R) - S\|_F < 2 \times 10^{-8}. \quad (\text{S27})$$

For an exact stationary point, concavity would imply global optimality; the finite residual in Eq. (S27) is a numerical stationarity check, not a rigorous objective-gap enclosure. If $\Omega = \sum_i \lambda_i |u_i\rangle\langle u_i|$, the recovered stationary-candidate PVM has $p_i = \langle u_i | R | u_i \rangle$, $q_i = \langle u_i | S | u_i \rangle$. We independently verify $\lambda_i = p_i/q_i$ and $\mathcal{F}(\Omega) = \sum_i p_i \ln(p_i/q_i)$ within the stated numerical tolerances.

The production grid contains 151 values of $\delta \in [0.20, 0.95]$ at each $\kappa \in \{0.70, 0.90, 0.99\}$. Pointwise, the optimized one-copy value agrees with the analytic expression above within 1.7×10^{-16} , and

$$\Sigma_M^{(1)} \leq \widehat{\Sigma}_{M,\text{num}}^{(2)} \leq D(\rho_\tau \| \pi_f), \quad \widehat{\Delta}_{\text{rot}} \leq \widehat{\Delta}_{\text{num}}. \quad (\text{S28})$$

The largest original-variable stationarity residual is 3.5×10^{-9} , the largest $|\lambda_i - p_i/q_i|$ residual is 5.9×10^{-9} , and the spectral-PVM value agrees with the variational value within 1.6×10^{-15} . A symmetry-reduced CVXPY 1.8.2/Clarabel audit at five corner and representative points uses a semidefinite quadrature approximation and agrees with the production values within 3.6×10^{-10} nats per copy.

On the displayed finite grid, the largest numerically recovered stationary-candidate gain is

$$\widehat{\Delta}_{\text{num}} = 0.0214758723 \text{ nats/copy} \quad \text{at} \quad (\delta, \kappa) = (0.810, 0.99). \quad (\text{S29})$$

At this point $\Sigma_M^{(1)} = 0.3791913303$, $\widehat{\Sigma}_{M,\text{num}}^{(2)} = 0.4006672026$, and $D = 0.5584510295$. The pair rotation attains 0.0073100901 nats per copy there. Across the full grid it captures 24.3%–49.4% of the recovered gap. Equation (S25) gives $N_5 = 38349.7$ rounds per arm in the continuous asymptotic calculation for the recovered stationary-candidate PVM at the best grid point; rounding upward gives 38350 rounds per arm and 115050 signal copies.

VIII. NUMERICAL METHODS AND REPRODUCIBILITY

The script `code/optimize_two_copy_sdp.py` generates `data/two_copy_sdp.csv` and `data/two_copy_sdp_validation.json`. Its analytic and primary-figure environment is Python 3.11.9 with NumPy 2.2.6, SciPy 1.17.0, Matplotlib 3.10.8, and CVXPY 1.8.2. The separately routed archived backend-snapshot environment is Python 3.11.9 with NumPy 2.4.6, SciPy 1.17.1, Matplotlib 3.11.1, Qiskit 2.5.2, Qiskit IBM Runtime 0.47.0, and Qiskit Aer 0.17.2. The project bundle preserves one disjoint lock for each route and records the actual runtime versions in `metadata/environment-provenance.yaml`. The filename `optimize_two_copy_sdp.py` records the initial semidefinite-program request; its production values use the exact log-coordinate objective in Eq. (S26), because CVXPY’s `quantum_rel_entr` is itself a quadrature approximation.

The companion script `code/simulate_qubit_activation.py` optimizes the explicit rotation over $0 \leq \theta \leq \pi/2$ using a deterministic 4097-point grid, bounded refinement of every sampled local maximum, and endpoint comparison.

A seeded audit checks 80 faithful pairs; the largest slope residual is 4.21×10^{-9} . A separate matrix-level implementation constructs the tensor-product states and rotated projectors without calling the analytic probability routine. Across 50 additional seeded cases, its largest probability residual is 1.67×10^{-16} and its largest slope residual is 2.65×10^{-9} . The files are `data/validation.json`, `code/independent_qubit_audit.py`, and `data/independent_validation.json`. The theorem proof is analytic and independent of these numerical checks.

-
- [S1] S. Deffner and E. Lutz, *Physical Review Letters* **105**, 170402 (2010).
 - [S2] G. T. Landi and M. Paternostro, *Reviews of Modern Physics* **93**, 035008 (2021).
 - [S3] Y. Li, V. Y. F. Tan, and M. Tomamichel, *Communications in Mathematical Physics* **392**, 993 (2022), arXiv:2104.14706 [quant-ph].
 - [S4] M. Berta, O. Fawzi, and M. Tomamichel, *Letters in Mathematical Physics* **107**, 2239 (2017).
 - [S5] A. W. van der Vaart, *Asymptotic Statistics* (Cambridge University Press, 1998).
 - [S6] IBM Quantum, Fake provider: Qiskit ibm runtime documentation, <https://quantum.cloud.ibm.com/docs/en/api/qiskit-ibm-runtime/fake-provider> (2026), accessed August 19, 2026.