

Minimum Bath Dimension for Qutrit Cooling Maps

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(Dated: 27 August 2026)

At low temperature, cooling maps and thermal operations have identical transition power for a two-level system. For qutrits, we introduce a rank-two cooling-map family Φ_a , where a mixes two coherent phase unitaries, and determine the minimum finite Gibbs-bath dimension required to implement it within diamond error ε . At cold-limit error 10^{-12} , $a = (\sqrt{5} - 1)/2$ forces $D \geq 128,000$, whereas $a = 1/2$ has an exact four-dimensional construction. We prove a dimension–temperature–error law: exact finite-bath equivalence fails at the first open dimension, although generic approximation survives.

Introduction.— Thermal operations model quantum control by an equilibrium bath and a fixed energy-conserving unitary. Their population and coherence restrictions are a foundation of quantum thermodynamic resource theories [1–7]. Finite reservoirs add size-dependent corrections to erasure, state conversion, cooling, and time–accuracy tradeoffs [8–14], while universal process implementations expose work, coherence, and control-complexity costs [15–18]. At low temperature, cooling maps—channels with diagonal or strictly energy-lowering Kraus operators—give more tractable state-transition criteria and motivate the conjecture that cooling maps and thermal operations allow the same transitions [19]. Equivalence was proved for two-level systems, leaving the qutrit as the first open dimension.

We isolate the exact finite-bath question hidden by the low-temperature approximation. Let H_S and H_B be the system and finite-bath Hamiltonians on $\mathcal{H}_S$ and $\mathcal{H}_B$. After setting the lowest bath energy to zero, write $G := \ker H_B$, $g := \dim G$, and Π_G for the orthogonal projector onto G , so Π_G/g is the maximally mixed state on G . For a system operator X and a joint unitary U on $\mathcal{H}_S \otimes \mathcal{H}_B$, let Tr_B denote the partial trace over $\mathcal{H}_B$, and let $\mathbb{I}_S$ and $\mathbb{I}_B$ denote the identity operators on $\mathcal{H}_S$ and $\mathcal{H}_B$. We understand $H_S + H_B$ as $H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B$. An ideal-zero-temperature operation is

$$\mathcal{T}(X) := \text{Tr}_B \left[U \left(X \otimes \frac{\Pi_G}{g} \right) U^\dagger \right], \quad [U, H_S + H_B] = 0. \quad (1)$$

We denote this class by TO_{fin}^0 . It sharpens the bath state approximately supported on its ground-state subspace in Ref. [19]; the distinction matters. No externally biased classical randomizer is supplied: all branch randomness must be generated by the finite thermal ancilla.

For a qutrit, set $d := \dim \mathcal{H}_S = 3$ and choose energy eigenstates $|k\rangle$ such that $H_S = \sum_{k=0}^2 E_k |k\rangle\langle k|$ with $E_0 < E_1 < E_2$. For a branch weight $0 < a < 1$ —the probability assigned to the first phase unitary Z_+ —define

$$\Phi_a(X) := aZ_+XZ_+^\dagger + (1-a)Z_-XZ_-^\dagger. \quad (2)$$

Here $Z_+ := \text{diag}(1, 1, e^{i\pi/4})$ and $Z_- := \text{diag}(1, -1, e^{3i\pi/4})$. This rank-two cooling map fixes every energy-diagonal

state and obeys $\Phi_a(e^{-itH_S}Xe^{itH_S}) = e^{-itH_S}\Phi_a(X)e^{itH_S}$ for every operator X and real t . Let $\mathbb{I}_n$ be the n -dimensional identity (subscript omitted when clear), and put

$$\rho := \frac{\mathbb{I}_3}{6} + \frac{1}{2}|+\rangle\langle +|, \quad \sigma_a := \Phi_a(\rho). \quad (3)$$

Here $|+\rangle := (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$. For a linear map Λ on qutrit operators, define the diamond norm by

$$\|\Lambda\|_\diamond := \sup_{\substack{\mathcal{H}_R \simeq \mathcal{H}_S \\ X \neq 0}} \frac{\|(\Lambda \otimes \text{id}_R)(X)\|_1}{\|X\|_1}, \quad (4)$$

Here $\|X\|_1 := \text{Tr} \sqrt{X^\dagger X}$ is the trace norm, and id_R is the identity map on operators of a reference system R . Our channel error $\|\Phi - \Psi\|_\diamond$ is unhalved: no factor $1/2$ is included. In the closure statement below, “generic” means that the nonzero Bohr frequencies $E_k - E_j$ are distinct.

First main result.— For every irrational a , the transition $\rho \mapsto \sigma_a$ is feasible by a qutrit cooling map but not by any $\mathcal{T} \in \text{TO}_{\text{fin}}^0$. Nevertheless, every generic qutrit cooling map lies in the diamond-norm closure of TO_{fin}^0 . Thus, at the first dimension beyond the two-level theorem, the exact finite-bath question splits into inequivalence and, for generic qutrit spectra, approximate equivalence. Distinct nonzero Bohr frequencies prevent interference between lowering paths. The irrational state-transition obstruction itself needs only distinct energy levels.

The obstruction is elementary but rigid. Both states in Eq. (3) have three positive, unchanged populations and nonzero coherences. These facts force every lowering block of a putative thermal dilation to vanish on G . The remaining $g \times g$ unitaries V_0, V_1, V_2 have moments under $\tau_g(Y) := \text{Tr}(Y)/g$ equal to the matrix B whose entries are defined by $B_{jk} := \langle j|\Phi_a(|j\rangle\langle k|)|k\rangle$. Those moments force $\Pi := (\mathbb{I}_g + V_1)/2$ to be a projection with $\tau_g(\Pi) = a$, impossible for irrational a [20]. Conversely, every complex 3×3 correlation matrix (positive semidefinite with unit diagonal) is a mixture of rank-one phase matrices [21]. Let $P_{j|k}$ be the probability that level k exits at j . For a rank-one term with $q_k := P_{k|k}$ and phase z_k with $|z_k| = 1$, choose a normalized state $|0_B\rangle$ spanning a one-dimensional bath ground-state subspace and orthogonal

states $|e_{jk}\rangle$ of energy $E_k - E_j$. With $|k, 0_B\rangle := |k\rangle \otimes |0_B\rangle$, the energy-preserving isometry

$$|k, 0_B\rangle \mapsto \sqrt{q_k} z_k |k, 0_B\rangle + \sum_{j < k} \sqrt{P_{j|k}} |j, e_{jk}\rangle \quad (5)$$

extends shellwise to an energy-conserving unitary. Rational convex weights are encoded by integer multiplicities in the bath ground-state subspace; rational approximation then proves diamond closure. The Supplemental Material (SM) supplies the complete construction [22].

At nonzero temperature and error, what bath dimension hides this obstruction? Set $k_B = 1$. For inverse temperature $0 < \beta < \infty$, a lower bound $\Delta > 0$ on every nonzero bath energy, and unhalved diamond error $0 \leq \varepsilon \leq 2$, set $Z_B := \text{Tr}(e^{-\beta H_B})$ for the bath partition function and $\Gamma_B := e^{-\beta H_B} / Z_B$ for its Gibbs state. For nonempty excited spectrum above Δ , define $D_{\min}^{(a)}(\beta, \Delta, \varepsilon)$ as the minimum $D := \dim \mathcal{H}_B$ over finite H_B and energy-conserving U such that a bath prepared in Γ_B implements Φ_a within diamond error ε ; set it to $+\infty$ if no implementation exists. Every admissible implementation then satisfies

$$\dim \mathcal{H}_B = D \geq D_{\min}^{(a)}(\beta, \Delta, \varepsilon). \quad (6)$$

Set $u := e^{-\beta \Delta} + \varepsilon$.

Put $m(a) := \min\{a, 1 - a\}$ and $\kappa(a) := \inf_{q \in \mathbb{N}} q \|qa\|_{\mathbb{Z}}$, where $\|x\|_{\mathbb{Z}} := \min_{p \in \mathbb{Z}} |x - p|$. The target is badly approximable iff $\kappa(a) > 0$; let $b(a)$ bound its continued-fraction coefficients. We suppress the fixed arguments of $D_{\min}^{(a)}$ below.

Second main result.— For such a badly approximable a , the minimum in Eq. (6) obeys, for all sufficiently small u ,

$$c_-(a)u^{-1/2} \leq D_{\min}^{(a)} \leq c_+(a)u^{-1/2}, \quad c_-(a) := \frac{\sqrt{\kappa(a)}}{2 + 2\sqrt{2}}, \quad c_+(a) := \frac{2\sqrt{2}[b(a) + 1]}{\sqrt{m(a)}}. \quad (7)$$

For the golden conjugate $\alpha := (\sqrt{5} - 1)/2$, one may take $b(\alpha) = 1$ and $\kappa(\alpha) = \alpha^2$, so $c_-(\alpha) = 0.12799\dots$; baths whose ground-state degeneracies are Fibonacci denominators give the upper staircase. Rational $a = p/q$ in lowest terms instead has an exact construction with $D \leq 2q$. For any rational approximant p/q used in the two-subspace construction, define $D_{\text{flag}} := 2q$ as the total Hilbert-space dimension of the Gibbs-flag bath: it contains two q -dimensional energy subspaces. Thus D_{flag} records the dimension used by this explicit construction; the optimal $D_{\min}^{(a)}$ may be smaller.

Formalism.— The theorem behind both the exact limit and the quantitative law uses the truncated resistance

$$\kappa_D(a) := \min_{\substack{q \in \mathbb{N} \\ 1 \leq q \leq D}} q \|qa\|_{\mathbb{Z}} = \min_{\substack{q \in \mathbb{N}, \\ 1 \leq q \leq D \\ p \in \mathbb{Z}}} q^2 \left| a - \frac{p}{q} \right|. \quad (8)$$

For any D -dimensional Gibbs-bath thermal operation Φ satisfying the gap condition above,

$$\|\Phi - \Phi_a\|_{\diamond} \leq \varepsilon \implies \frac{\kappa_D(a)}{D^2} \leq \frac{1}{4} e^{-\beta \Delta} + (12 + 8\sqrt{2})\varepsilon. \quad (9)$$

Arbitrary bath degeneracies, resonances, population leakage, and finite excited energies are allowed.

For the proof, with $k, \ell \in \{0, 1, 2\}$ write $U = \sum_{\ell, k} |\ell\rangle\langle k| \otimes M_{\ell|k}$ and $A_k := M_{k|k}$, where each $M_{\ell|k}$ acts on $\mathcal{H}_B$. Energy conservation gives $A_k \in \{H_B\}' := \{Y : [Y, H_B] = 0\}$, and unitarity makes each A_k contractive. The Gibbs functional $\tau(Y) := \text{Tr}(\Gamma_B Y)$ is tracial on this commutant. Measuring population leakage gives $d_k := 1 - \tau(A_k^\dagger A_k) \leq \varepsilon/2$. For $P_k := (A_k^\dagger A_k)^{1/2}$, complete the polar factor shellwise to a unitary V_k , so $A_k = V_k P_k$ and

$$|\tau(V_k^\dagger V_j) - \tau(A_k^\dagger A_j)| \leq d_k + d_j. \quad (10)$$

Equation (10) puts the three unitary moments within $\delta \leq 2\varepsilon$ of target. Gauge $V_0 = \mathbb{I}$, set $u_1 := V_1$, $v := V_2$, and $w := (u_1 + i\mathbb{I})/\sqrt{2}$. With $\|Y\|_{2,\tau} := [\tau(Y^\dagger Y)]^{1/2}$, the target geometry implies

$$\|v - w\|_{2,\tau}^2 \leq (1 + 2\sqrt{2})\delta. \quad (11)$$

From Eq. (11), functional calculus with $\text{Re } u_1 := (u_1 + u_1^\dagger)/2$, $s := \text{sgn}(\text{Re } u_1)$, choosing either sign on its zero eigenspace, and $\Pi := (\mathbb{I} + s)/2$ then gives

$$|\tau(\Pi) - a| \leq (6 + 4\sqrt{2})\delta. \quad (12)$$

The projection commutes with H_B . If r is its rank on the ground-state subspace, whose degeneracy is g , the excited Gibbs tail obeys

$$\left| a - \frac{r}{g} \right| \leq (12 + 8\sqrt{2})\varepsilon + \frac{D - g}{g} e^{-\beta \Delta}. \quad (13)$$

Combining Eq. (12) with the tail bound, multiplying by g^2/D^2 , and using $g(D - g)/D^2 \leq 1/4$ proves Eq. (9). Since $\kappa_D(a) \geq \kappa(a)$ and $12 + 8\sqrt{2} = (2 + 2\sqrt{2})^2$, the lower half of Eq. (7) follows.

Conversely, let $q \in \mathbb{N}$ and $0 \leq p \leq q$. For a projection $\Pi \in \{H_B\}'$, two q -dimensional energy shells and the controlled unitary $U_\Pi := Z_+ \otimes \Pi + Z_- \otimes (\mathbb{I} - \Pi)$ realize $\Phi_{\tau(\Pi)}$. Equal rank p on both shells gives $\tau(\Pi) = p/q$ at every temperature and

$$\|\Phi_{p/q} - \Phi_a\|_{\diamond} = 2|p/q - a|. \quad (14)$$

Alternatively, a flag projection of rank p on the ground-state subspace, together with a full or empty excited-energy part, tunes the weight to a with a Boltzmann factor of order $|a - p/q|$. Write $t := e^{-\beta \Delta}$. Exact synthesis meets the gap condition once $q \geq [m(a)t]^{-1/2}$; the equal-rank construction is ε -accurate once $q \geq \sqrt{2/\varepsilon}$. If p_n/q_n

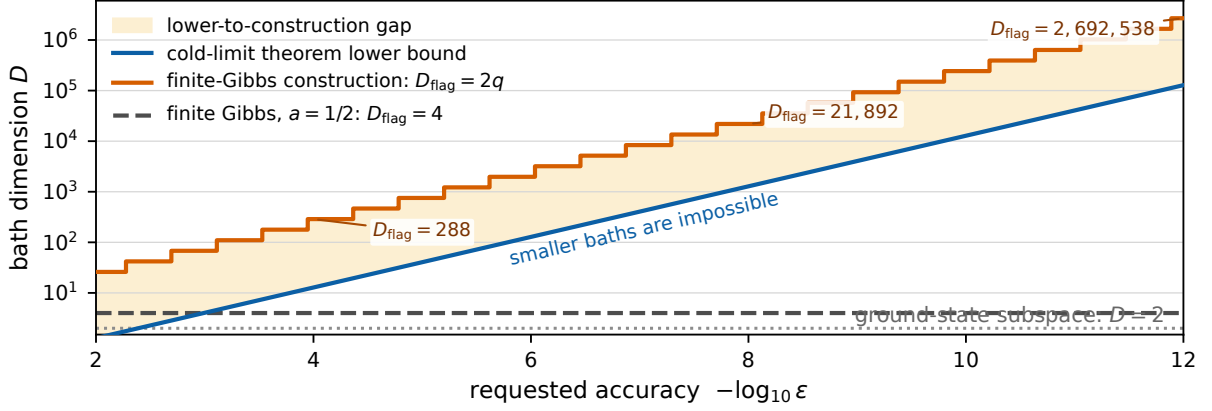


FIG. 1. Cold-limit comparison for $\alpha = (\sqrt{5} - 1)/2$. Blue is the theorem lower bound as $e^{-\beta\Delta} \rightarrow 0$; orange is the explicit finite-Gibbs D_{flag} staircase. Shading is the unresolved lower-to-construction gap, not $D_{\text{min}}^{(\alpha)}$. Dashed $D = 4$ is finite-Gibbs at $a = 1/2$; dotted $D = 2$ is for ground-state-subspace preparation only.

are the simple-continued-fraction convergents of a , then $|a - p_n/q_n| < q_n^{-2}$ and $q_{n+1} \leq [b(a) + 1]q_n$ [23],

$$D_{\text{min}}^{(a)} \leq \min \left\{ \frac{2[b(a) + 1]}{\sqrt{m(a)t}}, \frac{2\sqrt{2}[b(a) + 1]}{\sqrt{\varepsilon}} \right\} \quad (15)$$

$$\leq c_+(a)(t + \varepsilon)^{-1/2},$$

which proves the upper constant in Eq. (7).

The square-root law is exceptional as a uniform target-dependent law, but not as a typical finite-scale lower bound. The convergence half of Khintchine's theorem implies that, for every $\nu > 0$ and Lebesgue-almost every a , there is $A_\nu(a) > 0$ such that

$$\kappa_D(a) \geq \frac{A_\nu(a)}{[\log(2D)]^{1+\nu}} \quad (D \geq 2) \quad (16)$$

[24]. Equation (16) then gives, for sufficiently small u and Lebesgue-almost every a , there is $c_\nu(a) > 0$ such that

$$D_{\text{min}}^{(a)} \geq c_\nu(a) \frac{u^{-1/2}}{[\log(e/u)]^{(1+\nu)/2}}. \quad (17)$$

Irrationals outside the badly approximable class can still possess unusually cheap convergent subsequences; Liouville numbers, which admit rational approximations of arbitrarily high algebraic order, are an explicit extreme. Figure 1 makes the bath-dimension contrast explicit.

Applications.— Equation (9) gives an analytic bath-dimension exclusion test. Suppose a finite-sample learner returns $\hat{\Phi}$ and an upper diamond-radius r_U with stated confidence [25–29]. Set $\varepsilon_U := \|\hat{\Phi} - \Phi_a\|_\diamond + r_U$, an upper bound on the true error, and let independent thermometry and spectroscopy give a lower confidence bound $x_L \leq \beta\Delta$. On the joint confidence event, the actual ground-state degeneracy $1 \leq g \leq D - 1$ must satisfy the following

inequality, where κ_g is Eq. (8) with cutoff D replaced by g :

$$\kappa_g(a) \leq (12 + 8\sqrt{2})g^2\varepsilon_U + g(D - g)e^{-x_L}. \quad (18)$$

A candidate D is excluded when Eq. (18) fails for every allowed g . For the golden channel at $x_L = 10$, $\varepsilon_U < 1.8197 \times 10^{-3}$ excludes $D = 4$. Qutrit control and tomography are established in superconducting circuits [30, 31]; finite-calorimetry architectures have been studied separately [32], but no integrated implementation is demonstrated here. Such a test also requires calibrated Hamiltonians and joint dynamics, verified Gibbs preparation, a finite-sample process record, and confidence bounds for β and Δ .

Discussion.— The first result answers the exact finite-bath cooling-map question at its first open dimension: cooling maps reach strictly more exact qutrit transitions, yet both classes have the same closure for generic spectra. This qualification is essential because the original low-temperature argument used a bath only approximately supported on its ground-state subspace [19]. Equations (9) and (7) quantify the bath dimension needed to hide this inequivalence.

Previous finite-bath laws constrain cooling, heat and work corrections, energy bandwidth, time, or dissipation [8–13]. Our resource is instead the dimension needed to synthesize a coherent channel in diamond norm. The rational-trace seed is known in operator algebras [20, 33]; the thermodynamic state locking, qutrit closure, and dimension-gap-error law are the contributions developed here. Related separations between thermal operations and larger Gibbs-preserving or covariant channel classes have different mechanisms and resource questions [34, 35].

At $\varepsilon = 10^{-4}$, the temperature-independent construction has $D_{\text{flag}} = 4$ for $a = 1/2$ and $D_{\text{flag}} = 288$ for the golden conjugate; these costs need not equal D_{min} . At $\varepsilon = 10^{-12}$

the golden construction reaches $D = 2692538$, while the cold-limit theorem already forces $D \geq 128000$. Thus very large environments are unavoidable. Arbitrarily close rational and badly approximable targets can have bounded versus diverging minimum dimension as $e^{-\beta\Delta} + \varepsilon \rightarrow 0$.

The SM embeds Eq. (2) in a one-parameter rigid family and shows that its displayed member minimizes the proof coefficient at $12 + 8\sqrt{2}$.

Scope.— The law is model-specific. Free biased randomness removes the obstruction, while continuous baths, non-Gibbs preparation, approximate energy conservation, degenerate system energies, an upper bath-energy bound, or state-specific approximation may change it. The constants are not claimed optimal, and the complete $D_{\min}$ staircase outside the badly approximable class is open. Badly approximable weights have Lebesgue measure zero, but Eq. (17) supplies an almost-everywhere law with a logarithmic loss; a null exceptional set remains. The exclusion test is an ideal-model inference, not model-independent tomography.

Conclusion.— At the first dimension beyond the two-level theorem, the exact finite-bath convention yields inequivalence but generic closure. For the rigid qutrit family, convergence is a resource law coupling temperature, diamond error, bath dimension, and branch-weight arithmetic.

Research stewardship. Program Manager stewardship was provided by Domingos S. P. Salazar (research direction, physical scope, and project guardrails). This role is distinct from scientific authorship and from the separate release-authorization decision recorded in the associated Research Object.

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