

Supplemental Material for “Minimum Bath Dimension for Qutrit Cooling Maps”

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FROM THE TWO-LEVEL RESULT TO THE FIRST OPEN DIMENSION

Narasimhachar and Gour proved that cooling maps and low-temperature thermal operations coincide for a two-level system and conjectured equality of their state-transition power in arbitrary dimension [1]. Here we formulate the exact finite-bath limit used in the Letter and settle the first open system dimension.

Let H_S and H_B act on the system and finite-bath Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$, with $d := \dim \mathcal{H}_S = 3$. Shift the lowest bath energy to zero, set $G := \ker H_B$, let $g := \dim G$, and let Π_G be the orthogonal projector onto G , so Π_G/g is the maximally mixed state on G . For a system operator X and a joint unitary U on $\mathcal{H}_S \otimes \mathcal{H}_B$, let Tr_B denote the partial trace over $\mathcal{H}_B$, and let $\mathbb{I}_S$ and $\mathbb{I}_B$ denote the identity operators on $\mathcal{H}_S$ and $\mathcal{H}_B$. We understand $H_S + H_B$ as $H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B$. We write TO_{fin}^0 for the channels

$$\mathcal{T}(X) := \text{Tr}_B \left[U \left(X \otimes \frac{\Pi_G}{g} \right) U^\dagger \right], \quad [U, H_S + H_B] = 0. \quad (1)$$

For inverse temperature $\beta > 0$, the fixed-bath Gibbs state converges to Π_G/g as $\beta \rightarrow \infty$. Exact membership in this limiting class need not coincide with exact membership at any finite β . This is why the result below is an exact sharpening, rather than an unqualified refutation, of the physical low-temperature conjecture, whose bath state was only approximately supported on its ground-state subspace.

Choose qutrit energy eigenstates $|k\rangle$ such that $H_S = \sum_{k=0}^2 E_k |k\rangle\langle k|$ with $E_0 < E_1 < E_2$, and write $\rho_{jk} := \langle j|\rho|k\rangle$ for a density operator ρ . Here “generic” means that the nonzero Bohr frequencies $E_k - E_j$ are distinct. For a generic qutrit, a cooling map has diagonal Kraus operators together with energy-lowering Kraus operators proportional to $|j\rangle\langle k|$ for $j < k$. Its data are an upper-triangular column-stochastic matrix P , where $P_{j|k}$ is the probability for level k to exit at level j and $\sum_j P_{j|k} = 1$, and a positive semidefinite coherence matrix Q satisfying $Q_{kk} = P_{k|k}$. A cooling channel Φ acts as

$$[\Phi(\rho)]_{jj} = \sum_k P_{j|k} \rho_{kk}, \quad (2)$$

$$[\Phi(\rho)]_{jk} = Q_{jk} \rho_{jk} \quad (j \neq k). \quad (3)$$

We write $\mathbb{I}_n$ for the identity on an n -dimensional space and omit the subscript when its space is clear. For operators, $\|Y\|_1 := \text{Tr}[(Y^\dagger Y)^{1/2}]$ is the trace norm; for

linear maps, $\|\cdot\|_\diamond$ is the unhalved diamond norm. Distinct nonzero Bohr frequencies exclude additional path interference. The counterexample below needs only nondegenerate system energies because its population dynamics is the identity.

Exact qutrit separation by state locking

We first record the mechanism that converts one state transition into a full correlation-matrix constraint. Let ρ have three positive populations, and suppose an operation in TO_{fin}^0 maps it to a state with the same populations. Zero-temperature population flow is only downward. Starting at the highest level,

$$\rho_{22} = P_{2|2} \rho_{22}$$

forces $P_{2|2} = 1$ and removes the other entries in column 2. The middle level then gives $\rho_{11} = P_{1|1} \rho_{11}$, so column 1 is also trivial. Hence $P = \mathbb{I}_3$. If $L_{j|k}$ denotes the bath block of U for the transition $k \rightarrow j$, restricted to G , then $P_{j|k} = g^{-1} \text{Tr}(L_{j|k}^\dagger L_{j|k})$ and therefore vanishes by faithfulness of the matrix trace. If every input coherence is nonzero, Eq. (3) also fixes the whole matrix Q .

For $0 < a < 1$, set $\gamma := 2a - 1$ and

$$B_\gamma := \begin{pmatrix} 1 & \gamma & (\gamma - i)/\sqrt{2} \\ \gamma & 1 & (1 - i\gamma)/\sqrt{2} \\ (\gamma + i)/\sqrt{2} & (1 + i\gamma)/\sqrt{2} & 1 \end{pmatrix}. \quad (4)$$

With

$$z_+ := (1, 1, e^{i\pi/4})^T, \quad z_- := (1, -1, e^{3i\pi/4})^T,$$

direct multiplication gives $B_{2a-1} = az_+ z_+^\dagger + (1-a)z_- z_-^\dagger$. With $\odot$ denoting the entrywise (Schur) product, the map Φ_a defined by $\Phi_a(X) := B_{2a-1} \odot X$ is the cooling channel of Eq. (2) in the Letter. Define

$$|+\rangle := \frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}, \quad (5)$$

$$\rho := \frac{\mathbb{I}_3}{6} + \frac{1}{2}|+\rangle\langle +|, \quad \sigma_a := B_{2a-1} \odot \rho.$$

The input spectrum is $\{2/3, 1/6, 1/6\}$, both states have populations $(1/3, 1/3, 1/3)$, and every input coherence equals $1/6$. Hence $\rho \mapsto \sigma_a$ is cooling-map feasible.

Assume that an operation in TO_{fin}^0 realizes the same pair. State locking removes every lowering block. On the

bath ground-state subspace the dilation must therefore have, for every $|\psi\rangle \in G$, the form below, where $U(g)$ is the group of $g \times g$ unitary matrices:

$$U(|k\rangle \otimes |\psi\rangle) = |k\rangle \otimes V_k |\psi\rangle, \quad V_k \in U(g).$$

After a common gauge, set $V_0 = \mathbb{I}_g$ and write $\tau_g(Y) := g^{-1} \text{Tr}(Y)$. The three coherences in Eq. (4) impose

$$\tau_g(V_1) = \gamma, \quad (6)$$

$$\tau_g(V_2^\dagger) = (\gamma - i)/\sqrt{2}, \quad (7)$$

$$\tau_g(V_2^\dagger V_1) = (1 - i\gamma)/\sqrt{2}. \quad (8)$$

For $\|Y\|_{2,\tau_g} := [\tau_g(Y^\dagger Y)]^{1/2}$, put $W := (V_1 + i\mathbb{I}_g)/\sqrt{2}$. These identities give $\|W\|_{2,\tau_g} = 1$ and $\tau_g(V_2^\dagger W) = 1$, whence

$$\|V_2 - W\|_{2,\tau_g}^2 = 0. \quad (9)$$

Faithfulness yields $V_2 = W$. Unitarity of V_2 forces $V_1 = V_1^\dagger$, so

$$\Pi := \frac{\mathbb{I}_g + V_1}{2}, \quad \tau_g(\Pi) = \frac{1 + \gamma}{2} = a \quad (10)$$

is a projection. Its normalized trace is $\text{rank}(\Pi)/g$. For irrational a this is impossible. The same argument shows that the channel Φ_a itself is not in TO_{fin}^0 . The correlation matrix and its rational-trace obstruction originate in Ref. [2]; the thermodynamic state-locking step is what turns it into a counterexample to qutrit state reachability.

Every generic qutrit cooling map remains in the closure

Let (P, Q) be generic-qutrit cooling data. Put $q_k := Q_{kk}$ and discard indices with $q_k = 0$, whose rows and columns vanish by positivity. Then

$$C_{jk} := \frac{Q_{jk}}{\sqrt{q_j q_k}} \quad (11)$$

is a complex correlation matrix—a positive semidefinite matrix with unit diagonal—of dimension at most three. Every extreme such matrix has rank one [3]. For completeness, if an extreme C had rank $r \geq 2$, write it as the Gram matrix of three spanning unit vectors $v_k \in \mathbb{C}^r$. Hermitian $r \times r$ matrices form a real vector space of dimension $r^2 \geq 4$, whereas the conditions $v_k^\dagger H v_k = 0$ impose at most three real equations. A nonzero solution H makes $v_j^\dagger (\mathbb{I} \pm tH) v_k$ two distinct correlation matrices for small real t , contradicting extremality.

Finite-dimensional convexity therefore supplies probabilities $p_\ell \geq 0$ with $\sum_\ell p_\ell = 1$ and phase vectors $z^{(\ell)}$ such that

$$C = \sum_\ell p_\ell z^{(\ell)} z^{(\ell)\dagger}, \quad |z_k^{(\ell)}| = 1. \quad (12)$$

Extend every phase vector by arbitrary unit-modulus entries on the discarded indices; these entries are multiplied by $\sqrt{q_k} = 0$ and do not affect the channel. Each summand determines a rank-one cooling channel Φ_ℓ with the same P and an exact zero-temperature dilation. Suppress the label ℓ temporarily. Choose a normalized vector $|0_B\rangle$ in the bath ground-state subspace and, for every allowed transition $j < k$, a bath vector $|e_{jk}\rangle$ such that

$$H_B |0_B\rangle = 0, \quad H_B |e_{jk}\rangle = (E_k - E_j) |e_{jk}\rangle. \quad (13)$$

When released energies coincide, choose a sufficiently degenerate shell and take the corresponding flag vectors orthogonal. On $\text{span}\{|k, 0_B\rangle : k = 0, 1, 2\}$, where $|k, 0_B\rangle := |k\rangle \otimes |0_B\rangle$ and a comma abbreviates a tensor product, define

$$\mathcal{V} |k, 0_B\rangle := \sqrt{q_k} z_k |k, 0_B\rangle + \sum_{j < k} \sqrt{P_{j|k}} |j, e_{jk}\rangle. \quad (14)$$

Column stochasticity gives $q_k + \sum_{j < k} P_{j|k} = 1$. Hence every image has unit norm. Every term on the right of Eq. (14) has total energy E_k , while different k label distinct total-energy sectors. Thus $\mathcal{V}$ is an energy-preserving isometry. In each finite-dimensional total-energy eigenspace it extends to a unitary; the direct sum of these completions gives a finite U with $[U, H_S + H_B] = 0$. Tracing the bath returns the population matrix P and coherence multipliers

$$Q_{jk} = \sqrt{q_j q_k} z_j \bar{z}_k, \quad (15)$$

because all lowering flags are mutually orthogonal and orthogonal to $|0_B\rangle$.

If $p_\ell = n_\ell / L_{\text{mix}}$ are rational, with nonnegative integers n_ℓ summing to L_{mix} , choose an L_{mix} -dimensional ground-state subspace with basis $|m_B\rangle$, assign n_ℓ labels to branch ℓ , and for each transition choose flags $|e_{jk,m}\rangle$ at energy $E_k - E_j$, orthonormal over all transition-label pairs sharing a bath-energy shell. Apply Eq. (14) with $|0_B\rangle \mapsto |m_B\rangle$, $|e_{jk}\rangle \mapsto |e_{jk,m}\rangle$, and the phase vector assigned to label m . The resulting L_{mix} images in each total-energy sector are mutually orthogonal and again admit a shellwise unitary completion. The maximally mixed state on the ground-state subspace implements the weights n_ℓ / L_{mix} exactly. For arbitrary weights, choose rational probability vectors $\mathbf{p}^{(n)} := (p_\ell^{(n)})_\ell$ converging in ℓ_1 to $\mathbf{p} := (p_\ell)_\ell$; the resulting exact channels obey

$$\left\| \sum_\ell p_\ell^{(n)} \Phi_\ell - \sum_\ell p_\ell \Phi_\ell \right\|_\diamond \leq \sum_\ell |p_\ell^{(n)} - p_\ell| \rightarrow 0. \quad (16)$$

Thus every generic qutrit cooling channel belongs to $\overline{\text{TO}_{\text{fin}}^0}^\diamond$, the diamond-norm closure of TO_{fin}^0 . Exact attainment cannot replace closure, as Eq. (10) shows.

CHANNEL FAMILY AND COMPLETE THEOREM

Let $H_S = \sum_{k=0}^2 E_k |k\rangle\langle k|$ have three distinct energies. For $a \in (0, 1)$ put $\gamma := 2a - 1$ and define

$$B_\gamma := \begin{pmatrix} 1 & \gamma & (\gamma - i)/\sqrt{2} \\ \gamma & 1 & (1 - i\gamma)/\sqrt{2} \\ (\gamma + i)/\sqrt{2} & (1 + i\gamma)/\sqrt{2} & 1 \end{pmatrix}. \quad (17)$$

For

$$z_+ := (1, 1, e^{i\pi/4})^T, \quad z_- := (1, -1, e^{3i\pi/4})^T,$$

one verifies $B_{2a-1} = az_+z_+^\dagger + (1-a)z_-z_-^\dagger$. Hence the Schur channel Φ_a , defined by $\Phi_a(X) := B_{2a-1} \odot X$ and with $Z_\pm := \text{diag } z_\pm$, also has the explicit form

$$\Phi_a(X) = aZ_+XZ_+^\dagger + (1-a)Z_-XZ_-^\dagger. \quad (18)$$

This is Example 3.4 of Ref. [2]; its Proposition 3.2 gives the projection rigidity behind the finite-factorization obstruction. Here the whole interval of branch weights is retained because its arithmetic becomes a thermal resource. Since B_{2a-1} has unit diagonal, the channel is unital, fixes every energy-diagonal state, and is time-translation covariant.

Consider an arbitrary finite-bath thermal operation at inverse temperature β

$$\begin{aligned} \Phi(X) &:= \text{Tr}_B[U(X \otimes \Gamma_B)U^\dagger], \\ \Gamma_B &:= \frac{e^{-\beta H_B}}{\text{Tr } e^{-\beta H_B}}, \quad [U, H_S + H_B] = 0. \end{aligned} \quad (19)$$

We set Boltzmann's constant $k_B = 1$ and $0 < \beta < \infty$, shift the lowest bath energy to zero, write $D := \dim \mathcal{H}_B$, and assume that the nonempty excited spectrum is contained in $[\Delta, \infty)$ for a known $\Delta > 0$. The symbol Δ need only be a lower bound on nonzero bath energies, not the exactly known gap.

For $x \in \mathbb{R}$, let $\|x\|_Z := \min_{p \in \mathbb{Z}} |x - p|$. Define

$$\kappa_D(a) := \min_{\substack{q \in \mathbb{N} \\ 1 \leq q \leq D}} q \|qa\|_Z = \min_{\substack{q \in \mathbb{N}, 1 \leq q \leq D \\ p \in \mathbb{Z}}} q^2 \left| a - \frac{p}{q} \right|. \quad (20)$$

The two forms are identical term by term after minimizing over p . Set $m(a) := \min\{a, 1 - a\}$. If the unhalved diamond distance satisfies $\|\Phi - \Phi_a\|_\diamond \leq \varepsilon$ for $0 \leq \varepsilon \leq 2$, then

$$\frac{\kappa_D(a)}{D^2} \leq \frac{1}{4}e^{-\beta\Delta} + (12 + 8\sqrt{2})\varepsilon. \quad (21)$$

Taking $q = 1$ in Eq. (20) gives $\kappa_D(a) \leq m(a)$ unconditionally. No additional saturation branch is needed to prove Eq. (21); the parameter-uniform linear estimate suffices throughout.

For the actual error $\varepsilon_{\text{act}} := \|\Phi - \Phi_a\|_\diamond$ and $(y)_+ := \max\{y, 0\}$, Eq. (21) is equivalently

$$\varepsilon_{\text{act}} \geq \frac{1}{12 + 8\sqrt{2}} \left(\frac{\kappa_D(a)}{D^2} - \frac{1}{4}e^{-\beta\Delta} \right)_+. \quad (22)$$

For fixed $(H_S, \beta, \Delta, \varepsilon)$, let $D_{\min}^{(a)}(\beta, \Delta, \varepsilon)$ be the least finite D for which such an implementation exists, and set it to $+\infty$ if none exists. We suppress its fixed arguments below. The constants are valid but are not claimed optimal.

POPULATION LEAKAGE AND SHELLWISE UNITARY MOMENTS

Expand U in the system-energy basis, with $k, \ell \in \{0, 1, 2\}$ and bath operators $M_{\ell|k}$,

$$U = \sum_{\ell, k} |\ell\rangle\langle k| \otimes M_{\ell|k}, \quad A_k := M_{k|k}. \quad (23)$$

Energy conservation yields

$$H_B M_{\ell|k} - M_{\ell|k} H_B = (E_k - E_\ell) M_{\ell|k},$$

and therefore $A_k \in \mathcal{N} := \{Y : [Y, H_B] = 0\}$, the commutant of H_B . For Hermitian operators, write $Y_1 \preceq Y_2$ when $Y_2 - Y_1$ is positive semidefinite. From $U^\dagger U = \mathbb{I}$,

$$\sum_{\ell} M_{\ell|k}^\dagger M_{\ell|k} = \mathbb{I}, \quad A_k^\dagger A_k \preceq \mathbb{I}. \quad (24)$$

Hence each A_k is a contraction. Because Γ_B is scalar on each energy eigenspace,

$$\tau(X) := \text{Tr}(\Gamma_B X) \quad (25)$$

is a faithful tracial state on $\mathcal{N}$. We use the associated norm $\|Y\|_{2,\tau} := [\tau(Y^\dagger Y)]^{1/2}$.

For $\varepsilon_{\text{act}} := \|\Phi - \Phi_a\|_\diamond \leq 2$, on input $|k\rangle\langle k|$, the probability of returning to $|k\rangle$ is $p_k := \tau(A_k^\dagger A_k)$; define $d_k := 1 - p_k$. Applying the binary measurement $\{|k\rangle\langle k|, \mathbb{I} - |k\rangle\langle k|\}$ gives

$$2d_k \leq \|\Phi(|k\rangle\langle k|) - |k\rangle\langle k|\|_1 \leq \varepsilon_{\text{act}}. \quad (26)$$

No assumption about the other output populations is used.

Take the polar decomposition $A_k = W_k P_k$, where $P_k := |A_k| = (A_k^\dagger A_k)^{1/2}$. In each finite energy shell, extend the polar partial isometry to a unitary V_k ; hence

$$A_k = V_k P_k, \quad V_k \in \mathcal{N}. \quad (27)$$

This also covers singular and zero blocks. Put $\mu_k := \tau(\mathbb{I} - P_k) \leq \tau(\mathbb{I} - P_k^2) = d_k$. If C_0 is positive semidefinite ($C_0 \succeq 0$) and X_0 is a contraction in $\mathcal{N}$, tracial Cauchy-Schwarz implies

$$|\tau(C_0 X_0)| = |\tau(C_0^{1/2} X_0 C_0^{1/2})| \leq \tau(C_0). \quad (28)$$

For $\Omega_{kj} := V_k^\dagger V_j$, Eq. (28) yields the linear polar-mass estimate

$$\begin{aligned} |\tau(V_k^\dagger V_j) - \tau(A_k^\dagger A_j)| &= |\tau(\Omega_{kj} - P_k \Omega_{kj} P_j)| \\ &\leq |\tau[(\mathbb{I} - P_k) \Omega_{kj}]| + |\tau[P_k \Omega_{kj} (\mathbb{I} - P_j)]| \\ &\leq \mu_k + \mu_j \leq d_k + d_j. \end{aligned} \quad (29)$$

In the second term traciality moves $\mathbb{I} - P_j$ to the front.

Direct block multiplication fixes the moment orientation:

$$\langle j | \Phi(|j\rangle\langle k|) | k \rangle = \text{Tr}(A_j \Gamma_B A_k^\dagger) = \tau(A_k^\dagger A_j). \quad (30)$$

Testing the diamond norm on the unit-trace-norm operator $|j\rangle\langle k|$ bounds the difference from $(B_{2a-1})_{jk}$ by ε_{act} . Equations (26) and (29) therefore give unitaries $V_k \in \mathcal{N}$ such that

$$|\tau(V_k^\dagger V_j) - (B_{2a-1})_{jk}| \leq \delta, \quad \delta \leq 2\varepsilon_{\text{act}}. \quad (31)$$

PARAMETER-UNIFORM PROJECTION RIGIDITY

Apply the common left gauge $V_k \mapsto V_0^\dagger V_k$, write $u_1 := V_0^\dagger V_1$ and $v := V_0^\dagger V_2$, and retain $\gamma = 2a - 1$. Equation (31) implies

$$|\tau(u_1) - \gamma| \leq \delta, \quad (32)$$

$$\left| \tau(v) - \frac{\gamma + i}{\sqrt{2}} \right| \leq \delta, \quad (33)$$

$$\left| \tau(v^\dagger u_1) - \frac{1 - i\gamma}{\sqrt{2}} \right| \leq \delta. \quad (34)$$

Set $w := (u_1 + i\mathbb{I})/\sqrt{2}$. Since u_1 and v are unitaries,

$$\tau(w^\dagger w) = 1 + \text{Im } \tau(u_1), \quad \tau(v^\dagger w) = \frac{\tau(v^\dagger u_1) + i\tau(v^\dagger)}{\sqrt{2}}.$$

The second expression equals one at the target moments, independently of γ . Hence

$$\|v - w\|_{2,\tau}^2 \leq (1 + 2\sqrt{2})\delta. \quad (35)$$

Let $\text{Re } u_1 := (u_1 + u_1^\dagger)/2$ and $s := \text{sgn}(\text{Re } u_1)$, with either sign on a zero eigenspace, and $\Pi := (\mathbb{I} + s)/2$. With $|w| := (w^\dagger w)^{1/2}$, the polar unitary of w minimizes its tracial L_2 distance to the unitary group shell by shell, so

$$\||w| - \mathbb{I}\|_{2,\tau} \leq \|w - v\|_{2,\tau}. \quad (36)$$

For $z := e^{i\theta}$ and $y := \sin \theta$,

$$\begin{aligned} \frac{1 - |\cos \theta|}{(|(z + i)/\sqrt{2}| - 1)^2} &= \frac{(1 + \sqrt{1 + y})^2}{1 + |\cos \theta|} \\ &\leq (1 + \sqrt{2})^2 = 3 + 2\sqrt{2}, \end{aligned} \quad (37)$$

with removable zero-denominator points interpreted by continuity. Functional calculus and $|s - \text{Re } u_1| = \mathbb{I} - |\text{Re } u_1|$ now give

$$\begin{aligned} |\tau(s) - \gamma| &\leq \delta + \tau(\mathbb{I} - |\text{Re } u_1|) \\ &\leq [1 + (3 + 2\sqrt{2})(1 + 2\sqrt{2})]\delta \\ &= (12 + 8\sqrt{2})\delta. \end{aligned} \quad (38)$$

Therefore the projection expectation obeys, uniformly over a ,

$$|\tau(\Pi) - a| \leq (6 + 4\sqrt{2})\delta. \quad (39)$$

A nonisolated rigid phase family

The preceding phase pair minimizes the coefficient produced by the following estimate along a one-parameter family. For the phase parameter $\xi \neq 0$, put $\theta_\xi := \arctan \xi$ and

$$\begin{aligned} Z_+^{(\xi)} &:= \text{diag}(1, 1, e^{i\theta_\xi}), \\ Z_-^{(\xi)} &:= \text{diag}(1, -1, e^{i(\pi - \theta_\xi)}), \\ w_\xi &:= \frac{u_1 + i\xi \mathbb{I}}{\sqrt{1 + \xi^2}}. \end{aligned}$$

The three target moments are

$$\begin{aligned} \tau(u_1) &= \gamma, \quad \tau(v) = \frac{\gamma + i\xi}{\sqrt{1 + \xi^2}}, \\ \tau(v^\dagger u_1) &= \frac{1 - i\xi\gamma}{\sqrt{1 + \xi^2}}. \end{aligned}$$

Because $1 + i\xi = \sqrt{1 + \xi^2}e^{i\theta_\xi}$ and $e^{-i(\pi - \theta_\xi)}(i\xi - 1) = \sqrt{1 + \xi^2}$, one has $\tau(v^\dagger w_\xi) = 1$ at the target moments for every a . Define $\lambda := 2\xi/(1 + \xi^2)$; then

$$\tau(w_\xi^\dagger w_\xi) = 1 + \lambda \text{Im } \tau(u_1).$$

With moment error at most δ , put $c_\xi := 2(1 + |\xi|)/\sqrt{1 + \xi^2} + |\lambda|$; these identities imply

$$\|v - w_\xi\|_{2,\tau}^2 \leq c_\xi \delta. \quad (40)$$

Define $C_\xi := (1 + \sqrt{1 + |\lambda|})^2/\lambda^2$. The pointwise estimate needed for functional calculus is

$$1 - |\cos \theta| \leq C_\xi \left(\left| \frac{e^{i\theta} + i\xi}{\sqrt{1 + \xi^2}} - 1 \right| \right)^2. \quad (41)$$

Equations (40)–(41) reproduce Eq. (21) with $12 + 8\sqrt{2}$ replaced by

$$K_\xi := 1 + C_\xi c_\xi. \quad (42)$$

Since $(1 + |\xi|)^2/(1 + \xi^2) = 1 + |\lambda|$, both constants depend only on $|\lambda|$. Writing $h := \sqrt{1 + |\lambda|} \in (1, \sqrt{2}]$,

$$\begin{aligned} c_\xi &= 2h + h^2 - 1, \quad C_\xi = \frac{(1 + h)^2}{(h^2 - 1)^2} = \frac{1}{(h - 1)^2}, \\ C_\xi c_\xi &= \frac{h^2 + 2h - 1}{(h - 1)^2}, \end{aligned}$$

whose derivative $-4h/(h - 1)^3$ is negative on $h > 1$. Hence K_ξ is minimized at $|\lambda| = 1$, i.e., $|\xi| = 1$, where $K_1 = 12 + 8\sqrt{2}$.

GIBBS TAIL AND FINITE-DIMENSIONAL ARITHMETIC

Let g be the bath ground-state degeneracy and let r be the rank of Π on the ground-state subspace. For each excited energy $E > 0$, let P_E be its spectral projector and $d_E := \text{Tr}(P_E)$ its degeneracy. Write

$$R := \sum_{E>0} d_E e^{-\beta E}, \quad S := \text{Tr} \left(\Pi \sum_{E>0} e^{-\beta E} P_E \right).$$

Because $[\Pi, H_B] = 0$,

$$0 \leq S \leq R \leq (D - g)e^{-\beta \Delta}, \quad \tau(\Pi) = \frac{r + S}{g + R}. \quad (43)$$

It follows that

$$\left| \tau(\Pi) - \frac{r}{g} \right| = \frac{|gS - rR|}{g(g + R)} \leq \frac{D - g}{g} e^{-\beta \Delta}. \quad (44)$$

Combining with Eq. (39) and $\delta \leq 2\varepsilon_{\text{act}}$ gives

$$\left| a - \frac{r}{g} \right| \leq (12 + 8\sqrt{2})\varepsilon_{\text{act}} + \frac{D - g}{g} e^{-\beta \Delta}. \quad (45)$$

Since $g \leq D$, the definition of κ_D implies $g^2|a - r/g| \geq \kappa_D(a)$. Multiply Eq. (45) by g^2/D^2 , use $g^2/D^2 \leq 1$ and $g(D - g)/D^2 \leq 1/4$, and obtain Eq. (21).

The global Diophantine constant

$$\kappa(a) := \inf_{q \in \mathbb{N}} q \|qa\|_{\mathbb{Z}} = \inf_{\substack{q \in \mathbb{N} \\ p \in \mathbb{Z}}} q^2 \left| a - \frac{p}{q} \right| \quad (46)$$

satisfies $\kappa_D(a) \geq \kappa(a)$. It is positive exactly for badly approximable numbers, equivalently those with bounded continued-fraction coefficients [4]. The finite-scale constant is stronger for a specified D and remains useful when $\kappa(a) = 0$.

For the golden conjugate $\alpha := (\sqrt{5} - 1)/2$, first note that if $p/q \notin [0, 1]$, then $q^2|\alpha - p/q| \geq \min\{\alpha, 1 - \alpha\} = \alpha^2$. It therefore suffices to take $0 \leq p \leq q$. The other root of $x^2 + x - 1$ is $-1 - \alpha$; irrationality makes $p^2 + pq - q^2$ a nonzero integer, and hence

$$\begin{aligned} \left| \alpha - \frac{p}{q} \right| &= \frac{|p^2 + pq - q^2|}{q^2|p/q + 1 + \alpha|} \\ &\geq \frac{1}{q^2(2 + \alpha)} = \frac{\alpha^2}{q^2}. \end{aligned} \quad (47)$$

Equality in the infimum is already attained by $p/q = 1$, because $1 - \alpha = \alpha^2$. Thus $\kappa(\alpha) = \alpha^2$ and Eq. (22) reduces exactly to the golden-channel bound used in the preliminary theorem.

Almost-everywhere finite-scale resistance

The badly approximable class has Lebesgue measure zero, but a slightly weaker dimension law holds for Lebesgue-almost every branch weight. Fix $\nu > 0$ and set

$$\psi(q) := \frac{1}{q[\log(2q)]^{1+\nu}}. \quad (48)$$

Since $\sum_{q \in \mathbb{N}} \psi(q) < \infty$, the convergence half of Khintchine's theorem implies that, for Lebesgue-almost every a , the inequality $\|qa\|_{\mathbb{Z}} < \psi(q)$ has only finitely many solutions [5]. Absorbing those finitely many denominators into a positive target-dependent constant gives $A_\nu(a) > 0$ such that

$$q \|qa\|_{\mathbb{Z}} \geq \frac{A_\nu(a)}{[\log(2q)]^{1+\nu}} \quad (q \in \mathbb{N}). \quad (49)$$

Consequently,

$$\kappa_D(a) \geq \frac{A_\nu(a)}{[\log(2D)]^{1+\nu}}. \quad (50)$$

Let $u := e^{-\beta \Delta} + \varepsilon$ and $K := 12 + 8\sqrt{2}$. Equation (21) and $e^{-\beta \Delta}/4 + K\varepsilon \leq Ku$ give

$$D^2[\log(2D)]^{1+\nu} \geq \frac{A_\nu(a)}{Ku}. \quad (51)$$

If $D \geq u^{-1/2}$, the following bound is immediate. If $D < u^{-1/2}$, then $\log(2D) \leq \log(e/u)$ for all sufficiently small u ; inserting this into Eq. (51) proves

$$D_{\min}^{(a)} \geq c_\nu(a) \frac{u^{-1/2}}{[\log(e/u)]^{(1+\nu)/2}} \quad (52)$$

for Lebesgue-almost every a and all sufficiently small u , with for example $c_\nu(a) := \min\{1, \sqrt{A_\nu(a)/K}\}$. This is an almost-everywhere lower bound, not a matching characterization of the optimal staircase.

MATCHING CONSTRUCTIONS AND THE THREE CLASSES

Every projection $\Pi \in \mathcal{N}$ defines the energy-conserving controlled unitary

$$U_\Pi := Z_+ \otimes \Pi + Z_- \otimes (\mathbb{I} - \Pi). \quad (53)$$

Tracing a Gibbs bath gives $\Phi_{\tau(\Pi)}$: the flag weight is the projection expectation $\tau(\Pi)$. This converts rational approximation directly into thermal channel synthesis.

Let p/q , with $q \in \mathbb{N}$ and $0 \leq p \leq q$, be a convergent of a chosen close enough that the corresponding x below lies in $(0, 1)$. Give the bath a q -dimensional ground-state subspace and a q -dimensional excited-energy eigenspace

of energy $E > 0$. If $p/q < a$, let Π have rank p on the ground-state subspace and be the identity on the excited-energy eigenspace. Choosing

$$x := e^{-\beta E} = \frac{a - p/q}{1 - a} \quad (54)$$

gives $(p + qx)/(q + qx) = a$. If $p/q > a$, let Π have rank p on the ground-state subspace and vanish on the excited-energy eigenspace. The choice

$$x := e^{-\beta E} = \frac{p/q - a}{a} \quad (55)$$

gives $p/(q + qx) = a$. Thus convergents from either side produce exact implementations. On the lower branch, denote the perturbed branch weight by a_η . A small additive calibration error η in the dimensionless quantity βE sends $x = e^{-\beta E}$ to $x e^{-\eta}$ and changes the flag weight exactly by

$$a_\eta - a = \frac{(1 - a)x(e^{-\eta} - 1)}{1 + x e^{-\eta}} = -\frac{(1 - a)x}{1 + x} \eta + O(\eta^2).$$

Since the channel error is twice the flag-weight error, the symmetric sufficient condition

$$|\eta| \leq \log \left(1 + \frac{\varepsilon}{2(1 - a)x} \right) = \log \left(1 + \frac{\varepsilon e^{\beta E}}{2(1 - a)} \right) \quad (56)$$

keeps the implementation ε -accurate for both signs of η . If only $E \geq \Delta$ is known, replacing E by Δ gives a smaller conservative allowance. For a concrete golden lower branch, take the convergent $p/q = 144/233$. Then

$$x = \frac{\alpha - 144/233}{1 - \alpha}, \quad \beta E = -\log x = 10.7443685292 \dots$$

At $\varepsilon = 10^{-6}$, Eq. (56) gives $|\eta| \leq 0.0589259786 \dots$, or a fixed- β relative gap tolerance of $0.5484359 \dots \%$, safely 0.54% . This controls channel accuracy only. Preserving an independently established lower gap under negative η requires a nominal guard band or a post-calibration lower confidence bound. The approximation branch below requires no such tuning: $\tau(\Pi) = p/q$ at every temperature. Moreover

$$x \leq \frac{|a - p/q|}{m(a)} < \frac{1}{m(a)q^2}. \quad (57)$$

Taking $q \geq m(a)^{-1/2} e^{\beta \Delta/2}$ ensures $E \geq \Delta$.

For approximation, use the same two q -dimensional shells and give Π rank p on each. Then $\tau(\Pi) = p/q$ independently of E and temperature. Equation (18) gives

$$[\Phi_{p/q} - \Phi_a](X) = (p/q - a)(Z_+ X Z_+^\dagger - Z_- X Z_-^\dagger).$$

The difference of two channels has diamond norm at most two. Equality is attained here because Z_+ and Z_- send $(|0\rangle + |1\rangle)/\sqrt{2}$ to orthogonal states. Hence

$$\|\Phi_{p/q} - \Phi_a\|_\diamond = 2|p/q - a|. \quad (58)$$

Suppose now that a is badly approximable. If p_n/q_n are its convergents and $b(a)$ bounds all continued-fraction coefficients, then

$$\left| a - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}, \quad q_{n+1} \leq [b(a) + 1]q_n. \quad (59)$$

These standard continued-fraction facts are reviewed in Ref. [4]. Choose the first denominator exceeding $m(a)^{-1/2} e^{\beta \Delta/2}$ for exact synthesis, or the first exceeding $\sqrt{2/\varepsilon}$ for approximate synthesis. With $t := e^{-\beta \Delta}$, the bounded jump gives the explicit alternatives

$$D_{\min}^{(a)} \leq \min \left\{ \frac{2[b(a) + 1]}{\sqrt{m(a)t}}, \frac{2\sqrt{2}[b(a) + 1]}{\sqrt{\varepsilon}} \right\}. \quad (60)$$

Since $m(a) \leq 1/2$ and $\max\{t, \varepsilon\} \geq (t + \varepsilon)/2$, the right side is at most $c_+(a)(t + \varepsilon)^{-1/2}$ with $c_+(a) := 2\sqrt{2}[b(a) + 1]/\sqrt{m(a)}$.

Equation (21) and $\kappa_D(a) \geq \kappa(a) > 0$ give the reverse bound. Thus, with $u = e^{-\beta \Delta} + \varepsilon$ and $K = 12 + 8\sqrt{2} = (2 + 2\sqrt{2})^2$,

$$\frac{\sqrt{\kappa(a)}}{2 + 2\sqrt{2}} u^{-1/2} \leq D_{\min}^{(a)} \leq \frac{2\sqrt{2}[b(a) + 1]}{\sqrt{m(a)}} u^{-1/2} \quad (61)$$

for all sufficiently small u . The constants can be uniform only on subclasses bounded away from $a = 0, 1$, with a common upper bound on $b(a)$ and a common positive lower bound on $\kappa(a)$.

For the golden conjugate, $\sqrt{\kappa(\alpha)} = \alpha$ and the lower prefactor is

$$c_-(\alpha) := \frac{\alpha}{2 + 2\sqrt{2}} = \frac{\alpha(\sqrt{2} - 1)}{2} = 0.12799 \dots \quad (62)$$

If $a = p/q$ is rational in lowest terms, choose rank p on both shells. Equation (53) then realizes Φ_a exactly with $D = 2q$ for every β and every allowed Δ ; correspondingly, $\kappa_D(a) = 0$ once $D \geq q$. If a is irrational but not badly approximable, then $\kappa(a) = 0$ and there is no target-dependent positive quadratic constant. The finite theorem still applies, while the convergent constructions turn large ratios q_{n+1}/q_n into cost plateaus separated by cheap exceptional denominators. For a Liouville number, approximations $|a - p/q| < q^{-N}$ occur for arbitrarily large N , giving subsequences far cheaper than the square-root law. This establishes the three regimes claimed in the Letter without describing the badly approximable class as generic or claiming a complete formula for the optimal $D_{\min}$ staircase.

For $\alpha = (\sqrt{5} - 1)/2$, let $F_0 := 0$, $F_1 := 1$, and $F_{n+1} := F_n + F_{n-1}$ for $n \geq 1$ be the Fibonacci numbers. Its convergents are F_n/F_{n+1} and

$$\alpha F_{n+1} - F_n = (-1)^n \alpha^{n+1}. \quad (63)$$

Even and odd indices therefore supply the two exact constructions above, and the bounded denominator ratio is

the golden ratio. This is the explicit Fibonacci realization of Eq. (61).

At $\varepsilon = 10^{-4}$, the first rational flag satisfying $2|p/q - \alpha| \leq \varepsilon$ is $p/q = 89/144$. For any such approximant, define $D_{\text{flag}} := 2q$ as the total dimension of the explicit Gibbs-flag bath, which comprises two q -dimensional energy subspaces. Thus $D_{\text{flag}} = 288$ here. For $a = 1/2$, the same construction uses $D_{\text{flag}} = 4$. The finite search identifying the first denominator is included in `scripts/verify_theorem.py`. These are costs of this construction, not proofs of the globally minimal bath dimension.

FINITE-CONFIDENCE BATH-DIMENSION TEST

Nine linearly independent qutrit inputs may be chosen as the three energy eigenstates and, for each pair $j < k$, equal superpositions with relative phases 0 and $\pi/2$. Tomographically complete output measurements reconstruct the process matrix in the noiseless limit [6]; qutrit preparation, readout, and gate-set tomography are experimentally established [7, 8]. Informational completeness alone does not define a finite-sample confidence region. One may either equip this design with a rigorous confidence-region construction or use a direct diamond-distance channel-learning protocol [9].

Fix a before examining the data. Let $\hat{\Phi}$ be a channel estimate, let $\eta_{\Phi} \in (0, 1)$ be its allowed failure probability, and choose the upper radius r_U such that

$$\Pr[\|\Phi - \hat{\Phi}\|_{\diamond} \leq r_U] \geq 1 - \eta_{\Phi}.$$

A semidefinite program evaluates $\|\hat{\Phi} - \Phi_a\|_{\diamond}$ [10, 11], and the triangle inequality gives

$$\varepsilon_U := \|\hat{\Phi} - \Phi_a\|_{\diamond} + r_U.$$

Let $\eta_x \in (0, 1)$ be the calibration failure probability. Spectroscopy and thermometry should be summarized conservatively by a lower confidence bound $x_L \leq \beta\Delta$ on an event of probability at least $1 - \eta_{\Phi} - \eta_x$. By a union bound, both calibration and channel-learning statements hold with probability at least $1 - \eta_{\Phi} - \eta_x$, assuming $\eta_{\Phi} + \eta_x < 1$. Since $e^{-\beta\Delta} \leq e^{-x_L}$ and the true error is at most ε_U , define κ_g by Eq. (20) with cutoff D replaced by g . The true parameters (D, g) , with $1 \leq g \leq D - 1$, must satisfy for their actual ground-state degeneracy

$$\kappa_g(a) \leq (12 + 8\sqrt{2})g^2\varepsilon_U + g(D - g)e^{-x_L}. \quad (64)$$

A candidate dimension D is excluded when Eq. (64) fails for every $g \in \{1, \dots, D - 1\}$. Dividing by D^2 and using $g^2/D^2 \leq 1$ and $g(D - g)/D^2 \leq 1/4$ recovers the coarser closed form of Eq. (21). For $\kappa(a) > 0$ this implies

$$D \geq \frac{\sqrt{\kappa(a)}}{\sqrt{e^{-x_L}/4 + (12 + 8\sqrt{2})\varepsilon_U}}. \quad (65)$$

At negligible Gibbs tail, excluding dimension D requires accuracy of order $\kappa(a)/[(12 + 8\sqrt{2})D^2]$. For the golden channel at $x_L = 10$, the smallest exclusion threshold over $g = 1, 2, 3$ occurs at $g = D - 1 = 3$; the exact threshold is $1.819767739 \dots \times 10^{-3}$, so the safely truncated condition $\varepsilon_U < 1.8197 \times 10^{-3}$ excludes $D = 4$ on the joint confidence event.

This is an ideal-model bath-dimension test, not model-independent tomography. Exact energy conservation and Gibbs preparation must be separately calibrated. Confidence accounting must include state-preparation and measurement (SPAM) errors, drift, and model mismatch. The theorem does not cover a continuous bath, approximate energy conservation, non-Gibbs initial states, degenerate system eigenspaces, or approximation on only selected inputs. Free external biased randomness also removes the obstruction. Finally, no upper bath energy is imposed; an energy-bandwidth constraint introduces a fourth resource.

NUMERICAL AND REPRODUCIBILITY AUDIT

The Diophantine experiment was specified before its confirmatory runs. It tested four badly approximable weights: the golden conjugate, $\sqrt{2} - 1$, $\sqrt{3} - 1$, and the periodic continued fraction $[0; \overline{3, 1, 4}]$, where the overline denotes indefinite repetition of the block. The fitted exact-construction slopes in $\ln D$ versus $\beta\Delta$ were respectively 0.5000, 0.5000, 0.5000, 0.4956; the fitted approximation slopes in $\ln \varepsilon$ versus $\ln D$ were $-2.0000, -2.0000, -2.0000, -2.0116$. The maximum residual in the exact diamond identity, Eq. (58), was 2.19×10^{-16} .

As a falsification control, the Liouville constant $\mathcal{L} := \sum_{n \geq 1} 10^{-n!}$ was evaluated at 800-digit precision. Its truncated resistance fell to 10^{-480} at denominator 10^{120} , with effective approximation exponent $N_{\text{eff}} := -\log |\mathcal{L} - p/q| / \log q = 5.99749$, rejecting any weight-independent positive- κ or fixed-exponent interpretation. The displayed scans through $D = 10^6$ are validated without floating-point preselection. For every irrational target, exact rational enclosures share a continued-fraction prefix whose terminal denominator exceeds the cutoff; for the rational control, the exact-zero convergent terminates the search. The best-approximation theorem then restricts possible record minima of $q\|qa\|$ to those convergent denominators, whose value intervals are disjoint. Two final deterministic configurations, labeled 4b and 5 in the accompanying provenance log, produced eleven byte-identical outputs, including the timestamp-suppressed PDF. Earlier heuristic runs and a failed record-serialization run remain preserved in the provenance log. The frozen hash manifest in the data package identifies the principal figure, raw staircase table, results, and reproducibility comparison. These calculations test constructions and arithmetic segmentation; they are not used as proof.

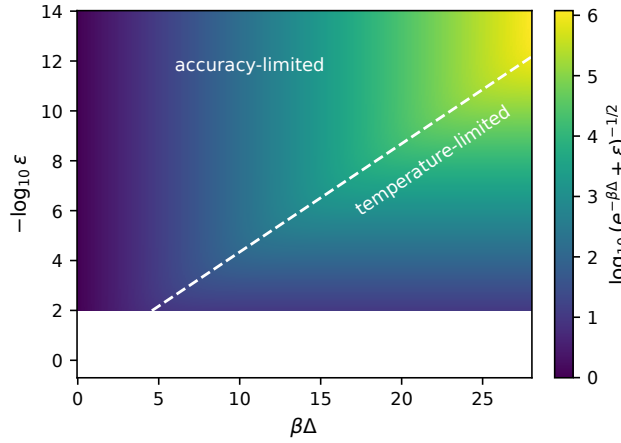


FIG. 1. For every fixed badly approximable target, the common scaling variable is $(e^{-\beta\Delta} + \epsilon)^{-1/2}$. The dashed line marks the crossover $\epsilon = e^{-\beta\Delta}$; target-dependent constants are not shown.

A separate pre-specified mechanism test sampled 60,000 triples (A_0, A_1, A_2) of contractions commuting with H_B in bath dimensions 2–8. No sample violated Eq. (29); the largest ratio of its left side to $d_k + d_j$ was 0.6747750361. A constructed rank-defect family, in which a polar block loses rank, attained ratio one from leakage 10^{-10} to 10^{-2} . Independent reruns matched all seven listed output hashes. The sampled contractions were not required to share one global energy-conserving completion, so this is a stress test over a conservative superset rather than physical evidence. The full reports, source, raw outputs, and frozen verification JSON accompany the data package.

Research provenance. Maestro v0.1 produced this private Pudim AI Research Ledger candidate through Codex-agent foraging, derivation, adversarial audit, numerical testing, and manuscript revision. Domingos S. P. Salazar supplied the research direction, physical scope, and Program Manager stewardship. Independent ChatGPT Oracle consultations and context-isolated PRL-style reviews were advisory. Exact deployment snapshots are unavailable and are not inferred; this disclosure grants no publication or release authority. The package-level role record is included with the source and reproducibility materials.

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