

Stochastic Thermodynamics of Highly Underdamped Homogeneous Hamiltonian Systems

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Abstract

Weak friction separates rapid Hamiltonian motion from slow thermal evolution. The fixed-control energy-diffusion operator organized by the enclosed phase volume $\Omega(E, \lambda)$ and density of states $\omega = \partial_E \Omega$ is an established part of Kramers theory and its multidimensional extensions. Here we place its driven form inside a density-of-states-consistent stochastic thermodynamics. Under energy lumpability, the averaged protocol drift, friction, and noise are fixed by $-\Omega_\lambda/\omega$ and Ω/ω ; the entropy $-k_B \int P \ln(P/\omega) dE$ then yields a first law, a trajectory entropy production Σ , and a nonequilibrium free-energy identity. Positive homogeneous Hamiltonians form an analytically closed class governed by $f_{\text{eff}} = 2\alpha$, which fixes state counting, heat capacity, and fluctuation amplitudes, and Γ_q , which fixes relaxation and power. For thermal exchange we recast the known gamma-difference law in a dimensionless moment-generating function controlled solely by $(\alpha, \Gamma_q t, \theta_0/T)$, and from it derive a compact all-order cumulant recurrence, an incomplete-beta violation probability, a heat-current uncertainty identity, skewness, and kurtosis. Its exact additive gap from the parabolic uncertainty benchmark is $1/\alpha$, although the multiplicative gap remains finite at fixed intensive driving. A control formulation separates continuous stroke optimization from discrete cycle architecture. A finite-time six-member atlas treats five named architectures and one smooth variational cycle on the same gamma manifold. The smooth cycle is the exact optimizer within the class of closed gamma-manifold loops having fixed period, mean virtual temperature, RMS mean-energy rate, and RMS relative thermal drive; its confinement waveform, heats, efficiency, entropy production, and covariance follow in closed form. The Otto reset limit has an exact work-uncertainty identity; Stirling contact allocation has a closed finite-time optimum; and Brayton pressure-ratio optimization recovers the Curzon–Ahlborn value under stated idealizations. Harmonic networks expose graph-dependent free energy, response, and size through spanning trees, resistances, and the Kirchhoff index. A collective graph-metric power potential preserves this entire geometry at arbitrary homogeneity degree while generating genuine many-body forces. For a sudden single-edge surgery we derive an exact beta–gamma work law controlled by the pre-quench effective resistance; it gives all work cumulants, Jarzynski and Crooks relations, the irreversible-entropy distribution, a joint heat–work transform after relaxation, and a graph-switching heat-engine cycle. For hard spheres we propose a conditional singular extension: assuming rapid shell mixing and orbit-averaged moving-wall work, the configurational integral acts as an effective control coordinate and the leading energy process remains CIR; packing then modifies physical compression through the compressibility factor and

graph-dependent excluded volume shifts Otto work statistics. A microscopic attractive tail generally breaks scalar closure, while a uniform mean field retains a nonequilibrium van der Waals form.

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I. INTRODUCTION

The low-friction limit of Brownian motion is one of the original regimes studied by Kramers. When a bound system completes many Hamiltonian orbits before friction appreciably changes its energy, phase becomes fast and energy becomes a slow stochastic coordinate [1–4]. Both the one-dimensional action equation and multidimensional Markovian energy diffusion written with the density of states and its integral are established results of reaction-rate theory [2, 5–7]. A particularly close density-of-states formulation is the multidimensional nonisothermal theory of Pollak, Talkner, and Berezhkovskii [8]. Slow external driving also has important antecedents: weakly damped periodically driven wells were reduced to energy–phase diffusion by Linkwitz and Grabert, while Jarzynski derived energy diffusion for slowly driven ergodic adiabatic ensembles and chaotic billiards [9–11]. Those works do not combine the present thermal bath current, deterministic phase-volume work transport, and DOS-relative stochastic-thermodynamic accounting, but they delimit the prior driven-energy-diffusion literature. That literature emphasizes escape and transport. Stochastic thermodynamics asks different questions of the reduction: heat and work distributions, entropy production, control, efficiency, and power.

Levitated particles made these questions analytically and experimentally concrete [12–15]. Salazar and Lira obtained the nonequilibrium heat law for a harmonic trap, including its exchange fluctuation relation and many-particle limit [16]. They then showed that $U(x) \propto x^{2n}$ preserves a gamma energy family, introduced fractional effective degrees of freedom, connected the steep limit to a particle in a box, and derived a minimum-work protocol with two thermally fast boundary transformations [17, 18]. The work moment-generating function was reduced to one scalar equation in Ref. [19]. Strong-detailed-fluctuation-theorem bounds on variance, apparent second-law violations, skewness, and moment-generating functions were developed in Refs. [20–23]; the broader entropy-production framework is reviewed in Ref. [24]. Chen *et al.* constructed a microscopic Curzon–Ahlborn engine and its joint work–heat statistics from the same energy process [25]. More recently, Chen and Quan optimized Brownian-engine power under stiffness and bath-switching constraints and obtained a distinct cycle containing two isotherms, two adiabats, and an additional isochoric relaxation [26].

We recover these established results as benchmarks. Our purpose is to expose their common geometry and locate a boundary of universality within the leading averaged, Markovian,

energy-lumpable framework. The organizing object is the phase volume enclosed by an energy shell. In one dimension it is the classical action; in higher dimensions it is the Hertz phase volume, with mixing providing one sufficient mechanism for microcanonical sampling [27, 28]. Its logarithmic energy slope defines

$$f_{\text{eff}}(E, \lambda) = 2E \partial_E \ln \Omega(E, \lambda). \quad (1)$$

For $U \propto x^{2n}$ this gives $f_n = (n + 1)/n$. A constant f_{eff} identifies a homogeneous universality class. A running value generally destroys the gamma closure and its scalar effective temperature.

We proceed from phase-space Langevin dynamics to energy-space thermodynamics and then solve the homogeneous class. The claim hierarchy is deliberately explicit. First, the fixed-control Ω/ω bath operator is established Kramers energy diffusion. Second, we synthesize established driven-energy-diffusion antecedents with thermal Kramers diffusion through the phase-volume continuity identity and DOS-relative entropy, thereby making the first law, stochastic entropy production, and nonequilibrium free energy mutually consistent under energy lumpability. Third, within the homogeneous class we recover the known gamma-difference heat and exchange-entropy laws, distinguish exchange entropy from total path entropy production, and derive the total-production distribution together with compact exchange cumulant, violation-probability, and uncertainty identities. Fourth, we distinguish continuous optimal control from imposed cycle topology, solve five named finite-time architectures, and obtain the exact smooth-loop optimizer within specified quadratic budgets. Fifth, we prove the common-rate closure and energy-fraction diffusion for independent subsystems and separate universal energy statistics from graph-dependent thermodynamic geometry. Sixth, a solvable collective graph interaction turns an effective resistance into the sole graph parameter of an arbitrary- q single-edge work law. The hard-sphere discussion develops a conditional driven singular extension in which the configurational integral becomes the control coordinate rather than silently treating excluded volume as another positive-homogeneous potential. The oscillator work and engine results serve as stringent recovery checks.

II. FROM LANGEVIN DYNAMICS TO ENERGY DIFFUSION

A. Exact energy balance

Consider

$$H(\mathbf{x}, \mathbf{p}; \lambda) = \sum_{i=1}^d \frac{p_i^2}{2m_i} + U(\mathbf{x}; \lambda) \quad (2)$$

with d quadratic momentum components coupled to a bath at temperature T . The underdamped Langevin equations are

$$\begin{aligned} dx_i &= \frac{p_i}{m_i} dt, \\ dp_i &= -\partial_{x_i} U dt - \gamma p_i dt + \sqrt{2m_i \gamma k_B T} dB_i. \end{aligned} \quad (3)$$

Itô's formula gives the exact microscopic first law

$$\begin{aligned} dE &= \left[\dot{\lambda} \partial_{\lambda} U + \gamma dk_B T - \gamma \sum_{i=1}^d \frac{p_i^2}{m_i} \right] dt \\ &\quad + \sqrt{2\gamma k_B T} \sum_{i=1}^d \frac{p_i}{\sqrt{m_i}} dB_i. \end{aligned} \quad (4)$$

The trajectory-level convention is $\delta W = \partial_{\lambda} H d\lambda$ and $\delta Q = dE - \delta W$, with both quantities positive into the system [29–31].

For one degree of freedom and a regular periodic orbit, define

$$J(E, \lambda) = \oint p dx, \quad \tau(E, \lambda) = \partial_E J(E, \lambda). \quad (5)$$

The identities $\overline{p^2/m} = J/J_E$ and $\overline{\partial_{\lambda} U} = -J_{\lambda}/J_E$ already show why thermal and protocol terms are geometric. The higher-dimensional version makes the structure more transparent.

B. Phase-volume identities

Let

$$\begin{aligned} \Omega(E, \lambda) &= \int d^d x d^d p \Theta[E - H(\mathbf{x}, \mathbf{p}; \lambda)], \\ \omega(E, \lambda) &= \partial_E \Omega(E, \lambda) \end{aligned} \quad (6)$$

be the enclosed phase volume and density of states. Microcanonical shell averages are normalized with $\delta(E - H)/\omega$. Differentiation with respect to the control gives

$$v_{\lambda}(E, \lambda) := \langle \partial_{\lambda} H \rangle_E = -\frac{\Omega_{\lambda}}{\omega}. \quad (7)$$

Momentum-space integration by parts gives

$$\left\langle \sum_{i=1}^d \frac{p_i^2}{m_i} \right\rangle_E = d \frac{\Omega}{\omega}. \quad (8)$$

Indeed, integrating $\partial_{p_i}\{p_i\Theta(E-H)\}$ equates one copy of Ω with the corresponding shell integral of p_i^2/m_i . This is the general phase-volume form of the virial ingredient used for power-law traps and applies beyond homogeneous Hamiltonians.

The same definitions imply

$$\partial_\lambda \omega + \partial_E(\omega v_\lambda) = 0. \quad (9)$$

This continuity identity states that Hamiltonian work transports density-of-states volume through energy space.

C. Averaged process

Fix a compact energy range $\mathcal{K}$ containing no separatrix or critical shell, and let $\tau_{\text{fast}}(E, \lambda)$ denote the recurrence or correlation time relevant to the required orbit averages. A convenient dimensionless control parameter is

$$\varepsilon = \sup_{E \in \mathcal{K}} \max \left\{ \gamma \tau_{\text{fast}}, |\dot{\lambda}| \tau_{\text{fast}} |\partial_\lambda \ln \Omega| \right\} \ll 1. \quad (10)$$

Choose a mesoscopic window satisfying

$$\tau_{\text{fast}} \ll \Delta t \ll \min\{\gamma^{-1}, t_{\text{drive}}, t_{\text{drift}}, t_{\text{diff}}\}. \quad (11)$$

In one dimension τ_{fast} is the orbital period. In several dimensions, mixing on a connected shell is a sufficient route to microcanonical averages. The precise requirement is *energy lumpability*: the orbit-averaged work coefficient, dissipative drift, and quadratic variation must depend on the slow state through E alone. Homogeneous virial identities and equal-rate sums provide important nonmixing examples.

Stochastic averaging [32, 33], including the quadratic variation in Eq. (4), gives the leading backward generator

$$\mathcal{L}_E \varphi = \left[-\dot{\lambda} \frac{\Omega_\lambda}{\omega} + d\gamma \left(k_B T - \frac{\Omega}{\omega} \right) \right] \varphi' + d\gamma k_B T \frac{\Omega}{\omega} \varphi''. \quad (12)$$

Corrections vanish in the averaging limit $\varepsilon \rightarrow 0$ on regular slow-time intervals; separatrices and accessible boundaries can require separate boundary-layer reductions. The associated Itô process is

$$dE = \left[-\dot{\lambda} \frac{\Omega_\lambda}{\omega} + d\gamma \left(k_B T - \frac{\Omega}{\omega} \right) \right] dt + \sqrt{2d\gamma k_B T \frac{\Omega}{\omega}} dB_E. \quad (13)$$

For one dimension, $\Omega = J$ and the fixed-control bath term is the action-based Kramers energy diffusion. The same Ω/ω structure is established in multidimensional energy-diffusion theory [2, 5, 8]. Energy diffusion caused by slow Hamiltonian driving is established for ergodic adiabatic ensembles and chaotic billiards, while weakly damped driven wells admit energy-phase reductions [9–11]. Equation (13) instead combines deterministic phase-volume work transport with thermal Kramers drift and diffusion after imposing energy lumpability; it is not claimed as the first driven energy-diffusion equation. Coefficients retaining additional conserved actions require a vector diffusion; additional actions compatible with energy-only coefficients still permit a closed marginal energy equation.

The density obeys $\partial_t P = -\partial_E j$ with

$$\begin{aligned} j &= j_w + j_b, & j_w &= \dot{\lambda} v_\lambda P, \\ j_b &= d\gamma \left(k_B T - \frac{\Omega}{\omega} \right) P - \partial_E (DP), & D &= d\gamma k_B T \frac{\Omega}{\omega}. \end{aligned} \quad (14)$$

At fixed parameters, zero bath current selects

$$P_{\text{eq}}(E|\lambda) = \frac{\omega(E, \lambda) e^{-E/k_B T}}{Z(\lambda, T)}, \quad Z = \int_0^\infty \omega e^{-E/k_B T} dE. \quad (15)$$

The density-of-states factor is essential: a flat Boltzmann density in E is correct only when ω is constant.

For applications that observe only finite energy cells, Appendix C gives a detailed-balance-preserving master equation whose states are classical bins.

III. ENERGY-SPACE STOCHASTIC THERMODYNAMICS

The thermodynamic entropy of an energy distribution must count the phase-space volume represented by each energy cell. Up to the fixed phase-space unit used in ω , it is

$$S[P, \lambda] = -k_B \int_0^\infty P(E, t) \ln \frac{P(E, t)}{\omega(E, \lambda)} dE. \quad (16)$$

The differential Shannon entropy $-k_B \int P \ln P$ omits the shell entropy and is unsuitable for thermodynamic balances when ω varies [34].

Assuming vanishing boundary currents, stochastic energetics defines [29, 30]

$$\dot{W} = \dot{\lambda} \int_0^\infty v_\lambda P \, dE, \quad \dot{Q} = \int_0^\infty j_b \, dE. \quad (17)$$

Integration by parts gives the first law

$$\frac{d \langle E \rangle}{dt} = \dot{W} + \dot{Q}. \quad (18)$$

Using Eq. (9), the protocol current transports the DOS entropy without producing it. The bath current has the gradient form

$$j_b = -DP \, \partial_E \ln \frac{P}{P_{\text{eq}}}, \quad (19)$$

and the stochastic entropy production along an energy trajectory $\mathcal{G}_{0:t} = \{E_s : 0 \leq s \leq t\}$ is the dimensionless path functional of stochastic thermodynamics [31, 35]

$$\Sigma_t[\mathcal{G}] = \frac{s_{\text{sys}}(E_t, t) - s_{\text{sys}}(E_0, 0)}{k_B} - \int_0^t \frac{\delta Q_s}{k_B T_s} = \ln \frac{\mathbb{P}_F[\mathcal{G}]}{\mathbb{P}_R[\mathcal{G}^\dagger]}. \quad (20)$$

Here

$$s_{\text{sys}}(E, t) = -k_B \ln \frac{P(E, t)}{\omega(E, \lambda_t)} \quad (21)$$

and the reverse ensemble begins from the forward final distribution. Equation (19) gives its mean production rate,

$$\frac{d}{dt} \langle \Sigma_t \rangle = \frac{1}{k_B} \left(\dot{S} - \frac{\dot{Q}}{T} \right) = \int_0^\infty \frac{j_b^2}{DP} \, dE \geq 0. \quad (22)$$

Equality with microscopic entropy production requires the eliminated phase variables to carry no additional irreversible current at the retained order. When hidden currents remain, Eq. (22) is a coarse-grained lower bound [34].

The nonequilibrium free energy

$$\mathcal{F}[P, \lambda, T] = \langle E \rangle - TS = F_{\text{eq}}(\lambda, T) + k_B T D_{\text{KL}}(P \| P_{\text{eq}}) \quad (23)$$

satisfies the general nonisothermal balance

$$\dot{\mathcal{F}} = \dot{W} - k_B T \frac{d}{dt} \langle \Sigma_t \rangle - S \dot{T}. \quad (24)$$

The last term is the change of the temperature multiplying the system entropy; it cannot be omitted on a nonisothermal stroke. For a fixed-temperature process beginning in equilibrium,

$$\frac{\langle W \rangle - \Delta F_{\text{eq}}}{k_B T} = \langle \Sigma_\tau \rangle + D_{\text{KL}}(P_\tau \| P_{\text{eq}, \tau}). \quad (25)$$

Consequently, at fixed T , $W_{\text{irr}} = k_B T \langle \Sigma_\tau \rangle$ when the final distribution is equilibrium. A nonequilibrium endpoint contributes the residual relative entropy in Eq. (25).

The entropy-production distribution is defined directly from the forward path measure:

$$P_F(\sigma, t) = \int \mathcal{D}\mathcal{G} \mathbb{P}_F[\mathcal{G}] \delta(\sigma - \Sigma_t[\mathcal{G}]). \quad (26)$$

Local detailed balance yields the forward–reverse detailed relation and its integral consequence [21, 31, 35],

$$\frac{P_F(\sigma, t)}{P_R(-\sigma, t)} = e^\sigma, \quad \langle e^{-\Sigma_t} \rangle = 1. \quad (27)$$

For a self-reversed ensemble the first identity becomes the strong detailed fluctuation theorem $P(\sigma)/P(-\sigma) = e^\sigma$. These statements concern the averaged path measure and reproduce the microscopic relations when hidden phase-space entropy production vanishes at the retained order. For isothermal forward and reverse reduced protocols initialized in equilibrium, the Crooks and Jarzynski relations follow [36, 37]:

$$\frac{P_F(W)}{P_R(-W)} = \exp\left(\frac{W - \Delta F}{k_B T}\right), \quad \langle e^{-W/k_B T} \rangle = e^{-\Delta F/k_B T}. \quad (28)$$

IV. PHASE-VOLUME HOMOGENEOUS UNIVERSALITY

Let the potential be positive, confining, and homogeneous of degree q in d coordinates,

$$U(\mathbf{x}; k) = k V_q(\mathbf{x}), \quad V_q(c\mathbf{x}) = c^q V_q(\mathbf{x}). \quad (29)$$

Scaling momenta by $E^{1/2}$ and coordinates by $(E/k)^{1/q}$ gives

$$\Omega(E, k) = C k^{-d/q} E^\alpha, \quad \alpha = \frac{d}{2} + \frac{d}{q}, \quad f_{\text{eff}} = 2\alpha = d \left(1 + \frac{2}{q}\right). \quad (30)$$

The prefactor C contains masses and the shape of V_q ; it drops out of normalized energy dynamics. Define

$$\lambda = k^{4/(q+2)}, \quad \ell = \sqrt{\lambda} = k^{2/(q+2)}, \quad g_q = \frac{2q}{q+2}, \quad \Gamma_q = \gamma g_q = \frac{d\gamma}{\alpha}. \quad (31)$$

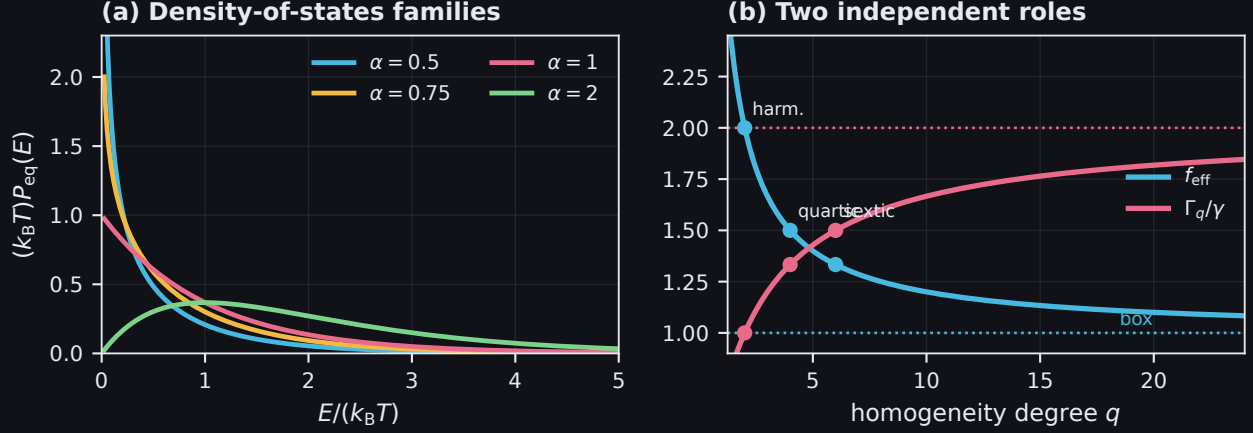


FIG. 1. Phase-volume universality. (a) Equilibrium gamma laws for representative $\alpha = f_{\text{eff}}/2$. (b) The one-dimensional power-law family separates $f_{\text{eff}} = 1 + 2/q$ from $\Gamma_q/\gamma = 2q/(q+2)$.

The symbol Γ_q is the energy-relaxation rate and agrees with the Salazar–Lira convention $\Gamma_n = 2n\gamma/(n+1)$ for $q = 2n$. The coordinate λ makes $\Omega \propto \lambda^{-\alpha/2} E^\alpha$ for every q .

Equation (13) becomes

$$\begin{aligned} dE &= [a(t)E - \Gamma_q(E - \alpha k_B T)] dt + \sqrt{2\Gamma_q k_B T E} dB_E, \\ a(t) &= \frac{1}{2} \frac{d}{dt} \ln \lambda(t) = \frac{\dot{\ell}}{\ell}. \end{aligned} \quad (32)$$

This driven Cox–Ingersoll–Ross process [38] displays the two class parameters cleanly. The exponent $\alpha = f_{\text{eff}}/2$ controls state counting, heat capacity, relative fluctuations, and cumulant amplitudes. The rate Γ_q controls relaxation, dissipation, and conversion from energy per cycle to power.

At fixed parameters,

$$\begin{aligned} P_{\text{eq}}(E) &= \frac{E^{\alpha-1} e^{-E/k_B T}}{\Gamma(\alpha) (k_B T)^\alpha}, \\ \langle E \rangle &= \alpha k_B T, & \text{Var } E &= \alpha (k_B T)^2, \\ C_V &:= \left(\frac{\partial \langle E \rangle}{\partial T} \right)_\lambda = \alpha k_B = \frac{f_{\text{eff}}}{2} k_B. \end{aligned} \quad (33)$$

Here C_V denotes heat capacity at fixed trap control; a mechanical volume can be included separately for a gas. Moreover, $Z = C\alpha\Gamma(\alpha)\lambda^{-\alpha/2}(k_B T)^\alpha$ and

$$\Delta F = \frac{\alpha k_B T}{2} \ln \frac{\lambda_f}{\lambda_i}. \quad (34)$$

TABLE I. Representative homogeneous classes.

System	α	f_{eff}	Γ_q/γ
1D harmonic, $q = 2$	1	2	1
1D quartic, $q = 4$	3/4	3/2	4/3
1D sextic, $q = 6$	2/3	4/3	3/2
1D hard box	1/2	1	2
d D power law	$d(1/2 + 1/q)$	$d(1 + 2/q)$	$2q/(q + 2)$

Conservation of phase volume during an isolated quasistatic stroke gives

$$E\lambda^{-1/2} = \text{constant}, \quad \theta\lambda^{-1/2} = \text{constant} \quad (35)$$

for a canonical family with virtual temperature θ .

A. Invariant thermal manifold

The driven process preserves the gamma family

$$P(E, t) = \frac{E^{\alpha-1} e^{-E/[k_B\theta(t)]}}{\Gamma(\alpha)[k_B\theta(t)]^\alpha}, \quad (36)$$

provided the initial state belongs to it. The virtual temperature obeys

$$\dot{\theta} = (a - \Gamma_q)\theta + \Gamma_q T. \quad (37)$$

On this manifold,

$$\dot{W} = \alpha k_B a \theta, \quad \dot{Q} = \alpha k_B \Gamma_q (T - \theta), \quad (38)$$

$$\frac{d}{dt} \langle \Sigma_t \rangle = \alpha \Gamma_q \frac{(\theta - T)^2}{\theta T}, \quad (39)$$

and

$$\mathcal{F} - F_{\text{eq}} = \alpha k_B T \left[\frac{\theta}{T} - 1 - \ln \frac{\theta}{T} \right]. \quad (40)$$

V. EXACT RESULTS WITHIN THE AVERAGED HOMOGENEOUS PROCESS

All results in this section are exact for Eq. (32). Their microscopic Langevin validity can become nonuniform: for one-dimensional $q > 2$, $\tau_{\text{fast}} \propto E^{1/q-1/2}$ diverges as $E \rightarrow 0$ while

$\alpha < 1$ makes the CIR origin accessible. Microscopic use therefore requires an energy range where Eq. (10) remains small, or a separate boundary-layer validation.

For fixed λ and bath temperature, let $r_t = e^{-\Gamma_q t}$. The transition density is

$$p(E, t|E_0) = c \exp[-c(E + r_t E_0)] \left(\frac{E}{r_t E_0} \right)^{(\alpha-1)/2} \times I_{\alpha-1} \left(2c \sqrt{r_t E E_0} \right), \quad c = \frac{1}{k_B T (1 - r_t)}. \quad (41)$$

Its first conditional moments are

$$\begin{aligned} \langle E_t | E_0 \rangle &= \alpha k_B T + (E_0 - \alpha k_B T) r_t, \\ \text{Var}(E_t | E_0) &= 2k_B T E_0 r_t (1 - r_t) + \alpha (k_B T)^2 (1 - r_t)^2. \end{aligned} \quad (42)$$

At equilibrium, $\text{Cov}(E_t, E_0) = \alpha (k_B T)^2 r_t$, and the relaxation modes are generalized Laguerre polynomials with rates $n\Gamma_q$. The origin is inaccessible for $\alpha \geq 1$ and accessible for $0 < \alpha < 1$. Thus one-dimensional quartic and steeper traps require a no-flux reflecting boundary. Orbit averaging is least uniform near this boundary.

A. Closed finite-time heat law

Prepare a gamma state with virtual temperature θ_0 and relax it at fixed λ in a bath at temperature T . Since $W = 0$, the absorbed heat is $Q = E_t - E_0$. Define

$$c_t = 1 - r_t, \quad \mu = k_B c_t (T - \theta_0), \quad b = k_B^2 c_t \theta_0 T. \quad (43)$$

The exact heat moment-generating function is

$$M_Q(s) = \langle e^{sQ} \rangle = (1 - \mu s - b s^2)^{-\alpha}. \quad (44)$$

The harmonic family of Ref. [16] has $\alpha = d$ for one particle and $\alpha = Nd$ for N independent particles; $\alpha = 1$ is its one-dimensional single-oscillator case. The nonharmonic energy family of Ref. [17] supplies fractional α . For any homogeneous dimension,

$$\begin{aligned} M_Q(s) &= (1 - \varepsilon_+ s)^{-\alpha} (1 + \varepsilon_- s)^{-\alpha}, \\ \varepsilon_{\pm} &= \frac{\sqrt{\mu^2 + 4b} \pm \mu}{2}. \end{aligned} \quad (45)$$

For $t > 0$, the transform exists on

$$-\varepsilon_-^{-1} < s < \varepsilon_+^{-1}; \quad (46)$$

at $t = 0$ the heat distribution is $\delta(q)$. Hence Q is the difference of two independent gamma variables of shape α and energy scales ε_+ and ε_- . The inverse transform is

$$p_t(q) = \frac{|q|^{\alpha-1/2} \exp\{\frac{q}{2}(\varepsilon_-^{-1} - \varepsilon_+^{-1})\}}{\sqrt{\pi} \Gamma(\alpha) \sqrt{\varepsilon_+ \varepsilon_-} (\varepsilon_+ + \varepsilon_-)^{\alpha-1/2}} \times K_{\alpha-1/2} \left[\frac{|q|}{2} (\varepsilon_+^{-1} + \varepsilon_-^{-1}) \right]. \quad (47)$$

The mean and variance are

$$\langle Q \rangle = \alpha \mu, \quad \text{Var } Q = \alpha(\mu^2 + 2b). \quad (48)$$

The heat law satisfies the exchange fluctuation relation [16, 39]. Our Q is positive into the system; Ref. [16] uses the opposite heat sign.

$$\frac{p_t(q)}{p_t(-q)} = \exp \left[\left(\frac{1}{k_B \theta_0} - \frac{1}{k_B T} \right) q \right]. \quad (49)$$

B. Wrong-way heat probability and large- N limit

For heating, a Clausius-sign event has $Q < 0$; for cooling, it has $Q > 0$. Let

$$z = \frac{1}{2} \left(1 - \frac{|\mu|}{\sqrt{\mu^2 + 4b}} \right). \quad (50)$$

The probability of such an apparently reversed heat event is

$$\mathcal{P}_{\text{wrong}} = I_z(\alpha, \alpha), \quad (51)$$

where I_z is the regularized incomplete beta function. It approaches 1/2 at short times, while at long times $z = \min(\theta_0, T)/(\theta_0 + T)$.

This event is an apparent violation of the macroscopic heat-flow direction. Negative total stochastic entropy production is a distinct event that also includes the system-entropy change. For a total-entropy-production variable obeying the strong detailed fluctuation theorem, Ref. [21] gives tight bounds on $\mathcal{P}(\Sigma < 0)$.

For N independent identical units, $\alpha_N = N\alpha_1$, all heat cumulants are extensive, and

$$- \lim_{N \rightarrow \infty} \frac{1}{N\alpha_1} \ln \mathcal{P}_{\text{wrong}} = - \ln[4z(1-z)]. \quad (52)$$

In the long-time limit, $M_Q(s) = [(1 - sk_B T)(1 + sk_B \theta_0)]^{-N\alpha_1}$. Heat in a single relaxation stroke is an endpoint difference, so it has no nonzero time-extensive scaled cumulant generating function. Long-time statistics require repeated cycles or continuous forcing.

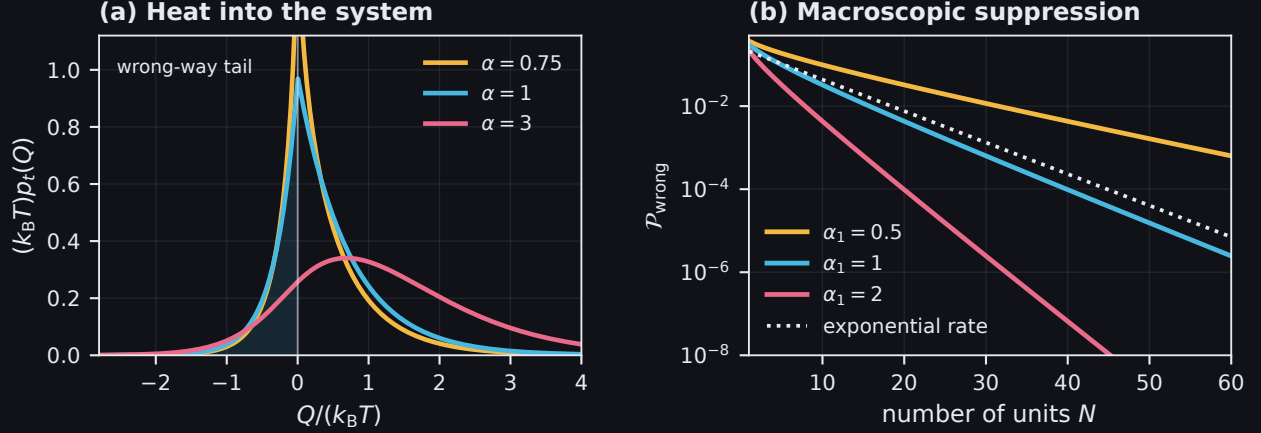


FIG. 2. Finite-time heat statistics. (a) Exact densities for heating from $\theta_0/T = 0.35$ at $\Gamma_q t = 1$. (b) The Clausius-sign probability from Eq. (51); the dotted line is its large- N rate approximation.

C. Closed entropy-production distribution

Equation (26) is a path integral for a general protocol. The fixed-control gamma relaxation admits a closed endpoint representation because

$$\theta_t = T + (\theta_0 - T)r_t, \quad \Sigma_t = c_\Sigma + A_t E_t + B_0 E_0, \quad (53)$$

where

$$c_\Sigma = \alpha \ln \frac{\theta_t}{\theta_0}, \quad A_t = \frac{T - \theta_t}{k_B \theta_t T}, \quad B_0 = \frac{\theta_0 - T}{k_B T \theta_0}. \quad (54)$$

Using the joint endpoint transform of the CIR process gives

$$M_\Sigma(z) := \langle e^{-z\Sigma_t} \rangle = e^{-z c_\Sigma} (1 + \ell_\Sigma z - m_\Sigma z^2)^{-\alpha},$$

$$\ell_\Sigma = \frac{\theta_0 - T}{T} (1 - r_t), \quad m_\Sigma = \frac{(\theta_0 - T)^2}{\theta_t T} r_t (1 - r_t). \quad (55)$$

Define the positive dimensionless scales

$$\rho_\pm = \frac{\sqrt{\ell_\Sigma^2 + 4m_\Sigma} \pm \ell_\Sigma}{2}. \quad (56)$$

Then, for $-\rho_+^{-1} < z < \rho_-^{-1}$, $M_\Sigma(z) = e^{-z c_\Sigma} (1 + \rho_+ z)^{-\alpha} (1 - \rho_- z)^{-\alpha}$, and equivalently $\Sigma_t \stackrel{d}{=} c_\Sigma + \rho_+ G_1 - \rho_- G_2$ for independent unit-scale gamma variables G_1, G_2 of shape α . For $x = \sigma - c_\Sigma$, the density is

$$P_\Sigma(\sigma; t) = \frac{|x|^{\alpha-1/2} \exp\{\frac{x}{2}(\rho_-^{-1} - \rho_+^{-1})\}}{\sqrt{\pi} \Gamma(\alpha) \sqrt{\rho_+ \rho_-} (\rho_+ + \rho_-)^{\alpha-1/2}} \\ \times K_{\alpha-1/2} \left[\frac{|x|}{2} (\rho_+^{-1} + \rho_-^{-1}) \right]. \quad (57)$$

The limiting degenerate cases follow by continuity. In particular, $P_\Sigma = \delta(\sigma)$ at $t = 0$ or $\theta_0 = T$. Writing P_θ for the gamma energy law at virtual temperature θ , the first two cumulants and the integral fluctuation theorem are

$$\begin{aligned}\langle \Sigma_t \rangle &= c_\Sigma + \alpha \ell_\Sigma = D_{\text{KL}}(P_{\theta_0} \| P_T) - D_{\text{KL}}(P_{\theta_t} \| P_T) \geq 0, \\ \text{Var } \Sigma_t &= \alpha(\ell_\Sigma^2 + 2m_\Sigma), \quad M_\Sigma(1) = 1.\end{aligned}\tag{58}$$

D. Strong-DFT exchange entropy

The endpoint entropy Σ_t above is the total stochastic entropy production of the one-way relaxation process. Its detailed theorem compares distinct forward and reverse ensembles through Eq. (27). The heat exchange relation (49) supplies a second, reversal-odd observable that obeys a strong DFT inside a single distribution:

$$\Sigma_x := \left(\frac{1}{k_B \theta_0} - \frac{1}{k_B T} \right) Q, \quad \frac{P_x(\sigma)}{P_x(-\sigma)} = e^\sigma.\tag{59}$$

This distinction fixes which uncertainty relations may be applied without importing a self-reversal assumption into $P_\Sigma(\sigma; t)$.

Introduce the single nonnegative nonequilibrium contrast

$$L = (1 - e^{-\Gamma_q t}) \frac{(T - \theta_0)^2}{T \theta_0} = (1 - r_t) \frac{(1 - u)^2}{u}, \quad u = \frac{\theta_0}{T}.\tag{60}$$

The Bessel- K gamma-difference distribution and its strong DFT are established for power-law-trap exchange entropy [21]; its generating-function and moment structure also belongs to the known gamma-difference/variance-gamma probability family [40, 41]. Here we reparametrize that family by the single contrast L . The formulas below collect the resulting all-order cumulant recurrence and exact gamma-family uncertainty and sign identities in the present stochastic-thermodynamic notation, together with their comparison to the distinct total production Σ_t .

The complete exchange-entropy statistics collapse to

$$\mathcal{G}_x(s) := \langle e^{s \Sigma_x} \rangle = [1 - Ls(1 + s)]^{-\alpha}, \quad \mathcal{G}_x(s) = \mathcal{G}_x(-1 - s).\tag{61}$$

The symmetry includes $\mathcal{G}_x(-1) = 1$. For $L > 0$, the moment-generating function exists on

$$-\frac{1 + \sqrt{1 + 4/L}}{2} < s < \frac{-1 + \sqrt{1 + 4/L}}{2},\tag{62}$$

and its cumulant-generating function is $K_x(s) = -\alpha \ln[1 - Ls(1 + s)]$. Define

$$\eta_{\pm} = \frac{\sqrt{L^2 + 4L} \pm L}{2}. \quad (63)$$

Then $\Sigma_x \stackrel{d}{=} \eta_+ G_1 - \eta_- G_2$ for independent unit-scale gamma variables of shape α . Its density is

$$\begin{aligned} P_x(\sigma) &= \frac{|\sigma|^{\alpha-1/2} e^{\sigma/2}}{\sqrt{\pi} \Gamma(\alpha) \sqrt{\eta_+ \eta_-} (\eta_+ + \eta_-)^{\alpha-1/2}} \\ &\quad \times K_{\alpha-1/2} \left[\frac{|\sigma|}{2} (\eta_+^{-1} + \eta_-^{-1}) \right]. \end{aligned} \quad (64)$$

The exponential tilt $e^{\sigma/2}$ displays the strong DFT directly.

E. Cumulants, apparent violations, and uncertainty gaps

For any random variable X , we define the cumulant-generating function and its cumulants by

$$K_X(s) := \ln \langle e^{sX} \rangle = \sum_{n=1}^{\infty} \frac{\kappa_n(X)}{n!} s^n, \quad \kappa_n(X) := \partial_s^n K_X(s) \big|_{s=0}. \quad (65)$$

Equation (61) then gives every exchange-entropy cumulant in one line:

$$\kappa_n(\Sigma_x) = \alpha(n-1)! \left[\eta_+^n + (-1)^n \eta_-^n \right]. \quad (66)$$

The first four reduce to polynomials in L ,

$$\begin{aligned} \kappa_1 &= \alpha L, & \kappa_2 &= \alpha L(L+2), \\ \kappa_3 &= 2\alpha L^2(L+3), & \kappa_4 &= 6\alpha L^2(L^2+4L+2). \end{aligned} \quad (67)$$

Writing $m = \langle \Sigma_x \rangle$ and $V = \text{Var } \Sigma_x$ eliminates L and exposes an exact hierarchy:

$$\begin{aligned} V &= 2m + \frac{m^2}{\alpha}, \\ \kappa_3 &= \frac{2m(V+m)}{\alpha} = \frac{6m^2}{\alpha} + \frac{2m^3}{\alpha^2}, \\ \kappa_4 &= \frac{6}{\alpha}(V^2 - 2m^2) = \frac{12m^2}{\alpha} + \frac{24m^3}{\alpha^2} + \frac{6m^4}{\alpha^3}. \end{aligned} \quad (68)$$

More generally, the two characteristic scales are roots of $r^2 - Lr - L = 0$. Their power-sum recurrence gives, for $n \geq 3$,

$$\kappa_n = (n-1)L\kappa_{n-1} + (n-1)(n-2)L\kappa_{n-2}. \quad (69)$$

Consequently, for $L > 0$, the signal-to-noise ratio, skewness, and excess kurtosis are

$$\begin{aligned} \frac{\langle \Sigma_x \rangle}{\sqrt{\text{Var } \Sigma_x}} &= \sqrt{\frac{\alpha L}{L+2}}, \\ \gamma_1 &= \frac{2\sqrt{L}(L+3)}{\sqrt{\alpha}(L+2)^{3/2}}, \quad \gamma_2 = \frac{6(L^2 + 4L + 2)}{\alpha(L+2)^2}. \end{aligned} \quad (70)$$

An even shorter hierarchy follows from the normalized squared signal-to-noise ratio

$$\chi := \frac{1}{\alpha} \frac{m^2}{V} = \frac{L}{L+2} = \frac{m}{m+2\alpha}, \quad 0 \leq \chi < 1. \quad (71)$$

The two non-Gaussian shape parameters are then simply

$$\gamma_1 = \frac{\sqrt{\chi}(3-\chi)}{\sqrt{\alpha}}, \quad \gamma_2 = \frac{3(1+2\chi-\chi^2)}{\alpha}. \quad (72)$$

These three dimensionless observables retain a finite far-from-equilibrium fingerprint: as $L \rightarrow \infty$, the signal-to-noise ratio tends to $\sqrt{\alpha}$, the skewness to $2/\sqrt{\alpha}$, and the excess kurtosis to $6/\alpha$. At $L = 0$, $P_x(\sigma) = \delta(\sigma)$; standardized statistics and noise-to-signal ratios are then undefined, and all $L \rightarrow 0$ statements below are limiting statements.

A negative exchange-entropy event, equivalently a wrong-way heat event, is $\Sigma_x < 0$. It is distinct from $\Sigma_t < 0$, the negative-total-entropy-production event. Indeed, with $c = 1 - r_t$ and $v = \theta_t/T = 1 + (u-1)r_t$, their means obey

$$\langle \Sigma_x \rangle = \alpha c \frac{(1-u)^2}{u}, \quad \langle \Sigma_t \rangle = \alpha \left[\ln \frac{v}{u} + (u-1)c \right]. \quad (73)$$

For $x = c(u-1)/u$, their difference is $\alpha[-x - \ln(1-x)] \geq 0$. For $L > 0$, define

$$z_x = \frac{1}{2} \left(1 - \sqrt{\frac{L}{L+4}} \right), \quad (74)$$

its probability and large-deviation rate per gamma-shape unit are

$$\begin{aligned} \mathcal{P}(\Sigma_x < 0) &= I_{z_x}(\alpha, \alpha), \\ - \lim_{N \rightarrow \infty} \frac{1}{N\alpha_1} \ln \mathcal{P}(\Sigma_{x,N} < 0) &= \ln \left(1 + \frac{L}{4} \right), \quad \alpha = N\alpha_1. \end{aligned} \quad (75)$$

Thus the probability, its macroscopic rate, and all standardized cumulants use only the usual suspects $(f_{\text{eff}}, \Gamma_q t, \theta_0/T)$. At $L = 0$, instead, $\mathcal{P}(\Sigma_x < 0) = 0$; Eq. (75) and its large- N rate assume fixed $L > 0$.

The exact noise-to-signal ratio is especially transparent:

$$\frac{\text{Var } \Sigma_x}{\langle \Sigma_x \rangle^2} = \frac{L+2}{\alpha L} = \frac{2}{\langle \Sigma_x \rangle} + \frac{1}{\alpha}. \quad (76)$$

For a random variable satisfying the strong DFT, the tight fluctuation-theorem TUR is [20]

$$\frac{\text{Var } \Sigma}{\langle \Sigma \rangle^2} \geq \text{csch}^2 \left[\frac{1}{2} h^{-1}(\langle \Sigma \rangle) \right], \quad h(a) = a \tanh \frac{a}{2}. \quad (77)$$

Equality requires the DFT-compatible binary distribution

$$P_a(\Sigma) = \frac{e^{a/2} \delta(\Sigma - a) + e^{-a/2} \delta(\Sigma + a)}{2 \cosh(a/2)}, \quad \langle \Sigma \rangle = a \tanh(a/2). \quad (78)$$

The same extremizer gives the tight sign bound [21]

$$\mathcal{P}(\Sigma < 0) \geq \frac{1}{2} \left[1 - \frac{\langle \Sigma \rangle}{h^{-1}(\langle \Sigma \rangle)} \right] \quad (79)$$

for a non-atomic law. More generally, the left side is $\mathcal{P}(\Sigma < 0) + \mathcal{P}(\Sigma = 0)/2$. The present gamma-difference family has an absolutely continuous, non-atomic law for finite $\alpha, L > 0$, so it cannot attain either binary equality condition. Its multiplicative gap from the tight TUR is

$$\mathfrak{G}_{\text{DFT}}(\alpha, L) = \frac{L + 2}{\alpha L} \sinh^2 \left[\frac{1}{2} h^{-1}(\alpha L) \right] > 1. \quad (80)$$

The familiar parabolic expression $2/\langle \Sigma_x \rangle$ also happens to be satisfied here, with the exact additive gap $1/\alpha$ and ratio

$$\frac{\text{Var } \Sigma_x / \langle \Sigma_x \rangle^2}{2/\langle \Sigma_x \rangle} = 1 + \frac{L}{2}. \quad (81)$$

The additive gap $1/\alpha$ vanishes as N^{-1} for N independent units, but this does *not* imply saturation at fixed intensive contrast L : the ratio in Eq. (81) remains $1 + L/2$. Saturation in the thermodynamic limit additionally requires $L \sim \alpha^{-1}$, so that the total mean m remains fixed while each unit approaches equilibrium. Near equilibrium, both the tight DFT bound and the parabolic benchmark reproduce the leading $2/m$ divergence. Far from equilibrium the tight binary bound becomes exponentially small and the parabolic benchmark misses the finite $1/\alpha$ noise floor. The tight DFT TUR remains universally valid; this gamma-difference family also happens to remain above the steady-current parabolic benchmark introduced in Refs. [42–45]. Both inequalities become diagnostically weak here, while the exact hierarchy, standardized shape, and sign probability retain distributional information [20, 22, 23, 46, 47].

There is also an important optimization caveat. At fixed α , Eq. (68) shows that reducing m reduces the *absolute* variance V , but the relative variance $V/m^2 = 2/m + 1/\alpha$ increases and the signal-to-noise ratio decreases. Moreover, Σ_x is a fixed-control exchange variable, whereas the minimum-work problem below minimizes the mean total pathwise production.

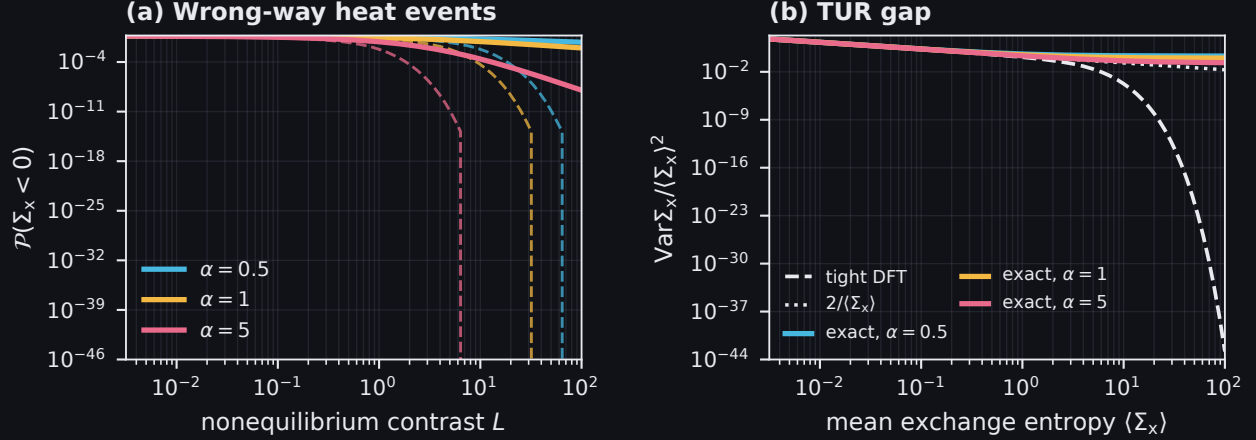


FIG. 3. Exchange-entropy statistics. (a) Exact wrong-way-heat probabilities (solid) and the binary strong-DFT floors (dashed). (b) The exact relative variance, the parabolic benchmark, and the tight binary-saturating DFT bound. The growing separation quantifies why a satisfied TUR can carry little far-from-equilibrium information.

Thus Eq. (76) does not by itself provide a second variational proof of the Carnot construction. A work identity can be inherited only under an additional trajectory-level tight-coupling relation, as demonstrated for the constrained Otto cycle below.

For thermal exchange the current form is especially clean. With affinity

$$\mathcal{A} := \frac{1}{k_B \theta_0} - \frac{1}{k_B T}, \quad \Sigma_x = \mathcal{A} Q, \quad (82)$$

the affinity cancels from the relative uncertainty, yielding the current–entropy–production identity

$$\frac{\text{Var } Q}{\langle Q \rangle^2} = \frac{2}{\langle \Sigma_x \rangle} + \frac{1}{\alpha}. \quad (83)$$

This is an exact finite-time heat-current identity for the gamma-difference family, not a generic current TUR.

VI. RECOVERY OF WORK AND JOINT HEAT–WORK STATISTICS

The next three sections use the phase-volume framework as an analytic consistency test against established oscillator results; they are retained in the main text to display the complete stochastic-thermodynamic chain from fluctuations to control and engines.

For a driven homogeneous system,

$$W = \int_0^\tau a(t)E_t dt, \quad Q = E_\tau - E_0 - W. \quad (84)$$

The affine structure reduces every transform to one Riccati equation. Define

$$\begin{aligned} G(t, E) &= \mathbb{E}_{E,t} \exp \left[-sE_\tau - z \int_t^\tau c(u)E_u du \right] \\ &= \exp[-\phi(t) - \psi(t)E]. \end{aligned} \quad (85)$$

The backward equation gives

$$\begin{aligned} \dot{\psi} &= (\Gamma_q - a)\psi + \Gamma_q k_B T \psi^2 - zc(t), & \psi(\tau) &= s, \\ \dot{\phi} &= -\Gamma_q \alpha k_B T \psi, & \phi(\tau) &= 0. \end{aligned} \quad (86)$$

For an initial gamma state with virtual temperature θ_0 ,

$$\left\langle \exp \left[-sE_\tau - z \int_0^\tau cE dt \right] \right\rangle = e^{-\phi(0)} [1 + k_B \theta_0 \psi(0)]^{-\alpha}. \quad (87)$$

Setting $c = a$ yields the work transform of Ref. [19]. The joint Laplace transform $\mathcal{M}_{Q,W}(u, z) = \langle e^{-uQ - zW} \rangle$ follows by solving Eq. (86) with terminal value $\psi(\tau) = u$ and functional coefficient $z - u$:

$$\mathcal{M}_{Q,W}(u, z) = e^{-\phi(0)} [1 + k_B \theta_0 (\psi(0) - u)]^{-\alpha}. \quad (88)$$

This transform is valid before Riccati blow-up and in the domain $1 + k_B \theta_0 [\psi(0) - u] > 0$ (with analytic continuation only where the corresponding expectation remains finite). Equations (86) and (88) recover the oscillator and engine transforms of Refs. [19, 25]; they belong to the broader counting-statistics treatment of stochastic work and engines [48]. The protocol changes the Riccati trajectory, Γ_q sets its clock, and α multiplies its cumulant generator.

A. When a work TUR exists

A work uncertainty relation is not obtained by merely renaming heat. For a one-bath driven process prepared in equilibrium, the forward–reverse path ratio identifies the dimensionless entropy production as

$$\Sigma = \beta(W - \Delta F), \quad \beta = (k_B T)^{-1}. \quad (89)$$

Accordingly, whenever the protocol and reverse ensemble give the strong DFT needed by Eq. (77), the dissipated work $W_{\text{diss}} = W - \Delta F$ inherits the same relative-variance relation. For a time-symmetric one-bath cycle, $\Delta F = 0$ and this is directly a work TUR. For a general protocol, however, Crooks' theorem compares distinct forward and reverse work distributions; a one-distribution strong DFT and its tight bound do not follow automatically [37, 49].

For a cyclic two-bath engine, take $Q_H > 0$ into the system and $W_{\text{out}} > 0$ delivered by it. Energy conservation and reservoir entropy give

$$\Sigma_{\text{cyc}} = Q_H(\beta_C - \beta_H) - \beta_C W_{\text{out}}. \quad (90)$$

Thus work alone is generally not proportional to entropy production: its uncertainty depends on the joint covariance of Q_H and W_{out} , which Eq. (88) can generate. If the joint antisymmetric current $\mathbf{X} = (Q_H, W_{\text{out}})^T$ obeys a strong DFT, with mean $\mathbf{q} = \langle \mathbf{X} \rangle$ and covariance C , the multidimensional form is

$$C - f(\langle \Sigma_{\text{cyc}} \rangle) \mathbf{q} \mathbf{q}^T \succeq 0, \quad f(x) = \text{csch}^2 \left[\frac{1}{2} h^{-1}(x) \right], \quad (91)$$

and in particular $\text{Var } W_{\text{out}} / \langle W_{\text{out}} \rangle^2 \geq f(\langle \Sigma_{\text{cyc}} \rangle)$ [20, 50]. This remains a bound. Only under trajectory-level tight coupling, $W_{\text{out}} = \eta Q_H$ with nonfluctuating η , does

$$\Sigma_{\text{cyc}} = \beta_C \left(\frac{\eta_C}{\eta} - 1 \right) W_{\text{out}}, \quad \eta_C = 1 - \frac{T_C}{T_H}, \quad (92)$$

so that a scalar entropy-production TUR becomes a work TUR. The finite-time Otto construction in Sec. VIII A provides an explicit realization; heat leaks, incomplete tight coupling, and nonadiabatic work fluctuations restore the multivariate or time-dependent problem [50, 51].

VII. RECOVERY OF MINIMUM WORK AND ORBIT-ADIABATIC BOUNDARY LAYERS

Consider an isothermal compression from λ_i to $\lambda_f > \lambda_i$ in time τ , beginning in equilibrium. Introduce

$$A = \frac{1}{2} \ln \frac{\lambda_f}{\lambda_i}, \quad g = \Gamma_q \tau. \quad (93)$$

The Euler–Lagrange equation has an exponential interior solution that cannot satisfy all endpoint conditions at finite g . Boundary layers are therefore required, as in earlier finite-time

optimal-control and irreversible-work problems [52–55]. Let $0 < \xi < 1$ be the unique root of

$$A = g\xi - 2\ln(1 - \xi). \quad (94)$$

The Salazar–Lira optimum [17] is

$$\begin{aligned} \frac{\lambda(0^+)}{\lambda_i} &= \frac{\lambda_f}{\lambda(\tau^-)} = (1 - \xi)^{-2}, \\ \lambda(t) &= \lambda(0^+)e^{2\Gamma_q \xi t}, \quad \theta(t) = \frac{T}{1 - \xi} \quad (0 < t < \tau). \end{aligned} \quad (95)$$

Its minimum work and irreversible part are

$$\begin{aligned} \frac{\langle W \rangle_{\min}}{\alpha k_B T} &= (1 - \xi)^{-2} - 1 + \frac{g\xi}{1 - \xi}, \\ W_{\text{irr}} &= \langle W \rangle_{\min} - \alpha k_B T A. \end{aligned} \quad (96)$$

The final layer leaves a nonequilibrium scale

$$\theta(\tau^+) = \frac{T}{(1 - \xi)^2}, \quad D_{\text{KL}}(P_{\tau^+} \| P_{\text{eq},f}) = \alpha \left[(1 - \xi)^{-2} - 1 + 2\ln(1 - \xi) \right]. \quad (97)$$

Thus the optimization constrains the final control value while leaving final equilibration free. Subsequent relaxation at fixed λ_f costs no work and dissipates precisely this residual nonequilibrium free energy.

The boundary transformation must respect the same scale separation that produced the energy diffusion. If τ_{layer} is its duration, the controlled regime is

$$\tau_{\text{fast}} \ll \tau_{\text{layer}} \ll \Gamma_q^{-1}. \quad (98)$$

Heat is negligible while many Hamiltonian periods still justify Eq. (35). A literal instantaneous stiffness quench is phase dependent and lies outside the scalar energy model. In this multiscale sense, variational calculus selects an isothermal exponential interior connected to two orbit-adiabatic layers.

VIII. CONSTRAINT-SELECTED OPTIMAL PROTOCOLS

The two jumps in Eq. (95) are one member of a larger control taxonomy. Write $y = \ln \ell$, so that $\dot{y} = a$. An ordinary-control regularization takes $a \in L^\infty[0, \tau]$, $a_- \leq a \leq a_+$, $\theta(0) = T$,

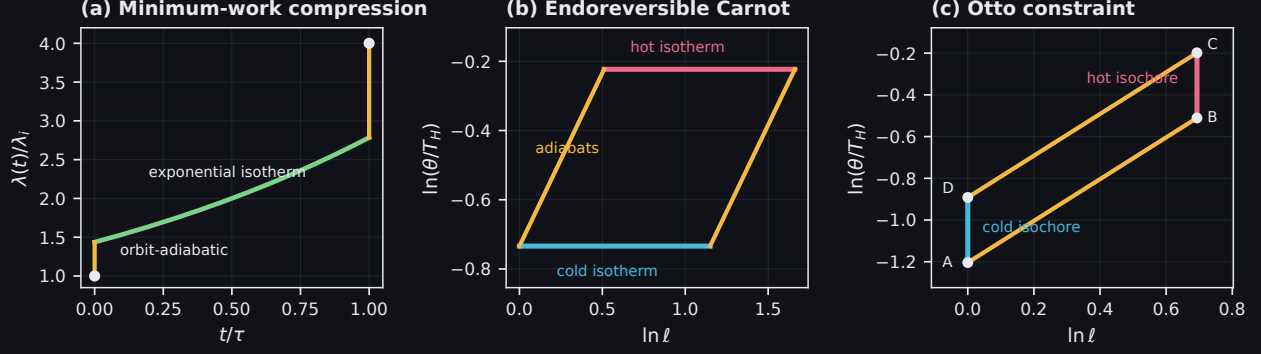


FIG. 4. Finite-time control. (a) The minimum-work isotherm has two orbit-adiabatic boundary layers and an exponential interior. (b) Two constant-internal-temperature isotherms joined by adiabats form the endoreversible Carnot construction. (c) Fixed-control heat contacts joined by orbit adiabats impose an Otto cycle.

fixed $y(0) = \ln \ell_i$ and $y(\tau) = \ln \ell_f$, and free terminal $\theta(\tau)$, hence the transversality condition $\pi(\tau) = 0$. On the thermal manifold the isothermal mean-work problem is

$$\min_a \langle W \rangle = \alpha k_B \int_0^\tau a \theta dt, \quad \dot{\theta} = (a - \Gamma_q) \theta + \Gamma_q T, \quad \dot{y} = a. \quad (99)$$

For the costate π conjugate to θ and the constant costate ν conjugate to y , the control Hamiltonian and switching function are

$$\begin{aligned} \mathcal{H} &= \alpha k_B a \theta + \pi [(a - \Gamma_q) \theta + \Gamma_q T] + \nu a, \\ \Phi &= \frac{\partial \mathcal{H}}{\partial a} = \theta (\alpha k_B + \pi) + \nu. \end{aligned} \quad (100)$$

The costate equation gives

$$\dot{\Phi} = \Gamma_q [T(\alpha k_B + \pi) - \alpha k_B \theta]. \quad (101)$$

On a singular arc, $\Phi = \dot{\Phi} = 0$. Since ν is constant, these conditions require

$$\theta = \theta_s = \text{constant}, \quad a_s = \Gamma_q \left(1 - \frac{T}{\theta_s} \right). \quad (102)$$

The generalized Legendre–Clebsch condition evaluates to

$$-\frac{\partial \ddot{\Phi}}{\partial a} = 2\Gamma_q \alpha k_B \theta > 0, \quad (103)$$

so the singular arc is locally minimizing. The exponential interior of Eq. (95) is therefore a singular isotherm in energy space. With finite rate bounds, standard Pontryagin theory

selects ordinary bang and singular arcs; the orbit-adiabatic endpoint layers of Sec. VII arise only after the large-bound limit is taken while preserving Eq. (98). This establishes the local extremal geometry, while global selection still requires feasibility and cost comparison.

Changing the admissible controls changes the geometry. For hard rate bounds $a_- \leq a \leq a_+$, the Hamiltonian is affine in a and the minimizing candidate obeys

$$a^*(t) = \begin{cases} a_-, & \Phi(t) > 0, \\ a_s, & \Phi(t) = 0 \text{ on an interval,} \\ a_+, & \Phi(t) < 0. \end{cases} \quad (104)$$

Depending on feasibility and switching signs, the solution may be bang–bang or bang–singular–bang. Bounds on stiffness can also make some endpoint pairs unreachable in a prescribed duration, as occurs in bounded Brownian-trap control [56]. General constrained-engine and limited-control formulations are developed in Refs. [57, 58]. If actuator effort is assigned the quadratic cost $(\chi/2) \int a^2 dt$, stationarity instead yields the finite feedback candidate

$$a_\chi^*(t) = -\frac{\Phi(t)}{\chi}, \quad a_- \leq a_\chi^* \leq a_+, \quad (105)$$

with clipping at the rate limits. The parameter χ regularizes the boundary layers; smoothness beyond continuity depends on the active-set transitions.

Several experimentally distinct objectives fit the same playbook. A hard heat-current limit constrains

$$|\dot{Q}| = \alpha k_B \Gamma_q |T - \theta| \leq J_{\max}, \quad (106)$$

and creates constant-flux boundary arcs. State bounds $\lambda_- \leq \lambda \leq \lambda_+$ create dwell arcs at the admissible trap edges. Requiring $\theta(\tau) = T$ changes the terminal transversality condition and removes the free residual relative entropy in Eq. (97). Minimum-time control with bounded a or bounded bath temperature is bang dominated because the controls enter the state equation affinely; related bang–bang thermal connection problems are discussed in Ref. [59]. A fluctuation-sensitive design can use

$$\mathcal{J}_{\chi, \zeta}[a] = \langle W \rangle + \frac{\chi}{2} \int_0^\tau a^2 dt + \zeta \text{Var } W, \quad \zeta \geq 0, \quad (107)$$

where Eq. (86) or an augmented moment system supplies the variance. The resulting extremal depends on the fluctuation costate and need not retain a constant- θ singular arc.

Cycle names become precise only after the admissible stroke manifolds are specified [60, 61]. On the gamma manifold their mean thermodynamic dictionary is

$$d\langle E \rangle = \alpha k_B d\theta, \quad \delta W = \alpha k_B \theta dy, \quad \delta Q = \alpha k_B (d\theta - \theta dy), \quad dS = \frac{\delta Q}{\theta}. \quad (108)$$

Thus $a = \dot{y} = 0$ selects an isochore, $\Gamma = 0$ selects an orbit adiabat, and $\dot{\theta} = 0$ during bath contact selects an energy-space isotherm. Otto admissibility is the complementarity condition that mechanical driving and bath contact never act simultaneously; Carnot admissibility permits simultaneous driving and contact on two prescribed branches. The remaining families require additional global or stroke constraints:

Carnot: bath-coupled isotherms and isolated orbit adiabats; the continuous optimization selects singular contact arcs and contact times within this architecture [25, 62, 63].

Otto: fixed- λ thermal contacts and isolated orbit-adiabatic changes; the finite-contact cycle and its compression ratio are solved below [64–67].

Stirling: bath-coupled isotherms and fixed- λ thermal switches; ideal recovery additionally requires a prescribed regenerator [68–70].

Ericsson: isotherms and isobars; a physical volume–pressure control and a regenerator must be supplied [71, 72].

Brayton–Joule: isobars and isolated adiabats; physical pressure control and its switching costs must be supplied [60, 73].

Smooth quadrature: simultaneous mechanical and thermal control over a fixed period, with prescribed mean virtual temperature, RMS mean-energy rate, and RMS relative thermal drive; the loop follows from the global optimization in Sec. IX F.

These constraints define the named architectures; they are not claims that unrestricted variational calculus selects their topology. Limited-control and global cycle optimization are distinct problems [57, 74–76].

The five textbook architectures have been unified previously at the quasistatic level, including arbitrary-order fluctuation relations for small heat engines [61]. The purpose of Fig. 5 is to hold the finite-time CIR reduction fixed, make the extra control constraints explicit, solve the corresponding periodic scale maps, and record which optimizations remain

closed. The sixth panel adds a continuous-control counterpart selected by global quadratic budgets. The atlas itself is organizational; finite-time identities are established cycle by cycle.

For comparison on one physical pressure–volume model, consider the homogeneous ideal-gas phase volume

$$\Omega(E, V) = CV^N E^\alpha, \quad \ell = \ell_0 \left(\frac{V}{V_0} \right)^{-N/\alpha}, \quad \langle p \rangle = \frac{Nk_B \theta}{V}. \quad (109)$$

It has

$$C_V = \alpha k_B, \quad C_P = (\alpha + N)k_B, \quad \gamma_{\text{ad}} = \frac{C_P}{C_V} = 1 + \frac{N}{\alpha}. \quad (110)$$

With $x = \ln(\ell/\ell_0)$ and $z = \ln(\theta/T_C)$, the four stroke slopes are

$$\left. \frac{dz}{dx} \right|_{T_\theta} = 0, \quad \left. \frac{dz}{dx} \right|_V = \infty, \quad \left. \frac{dz}{dx} \right|_S = 1, \quad \left. \frac{dz}{dx} \right|_p = -\frac{\alpha}{N}. \quad (111)$$

Here T_θ denotes constant virtual temperature, and the isobar is a constant ensemble-mean pressure. Isotropic dilation, or sufficient mixing to erase directional energies, is required for this scalar closure.

The six examples have different levels of stochastic closure. Carnot and Otto remain inside constant-coefficient CIR segments, and the Otto reset limit gives an elementary closed work law. Stirling is exact through a product of four fractional-linear Riccati maps. Ericsson and Brayton have exact gamma-manifold corner maps and mean thermodynamics; their fluctuating mean-isobar work is a time-integrated functional represented by a time-dependent Riccati propagator. The smooth quadrature cycle has an exact global mean optimum and becomes an autonomous CIR process after normalization and a deterministic time change; its integrated-work law still uses a time-dependent Riccati propagator. Thus the atlas compares admissible controls and exact mean optima without implying the same closed trajectory distribution for every architecture.

For a two-temperature engine, each bath-contact work extremal selects a constant virtual temperature through Eq. (102); Eq. (35) supplies a zero-heat connector. Consequently, within a four-stroke architecture containing one contact with each bath and two orbit-adiabatic connectors, stroke optimization recovers two endoreversible isotherms and Carnot geometry. The derivation selects the shape of each admissible stroke; it does not prove that an unrestricted global optimization over bath switching, stroke number, and ordering selects the four-stroke topology itself. Rate bounds replace jumps by finite bangs, finite

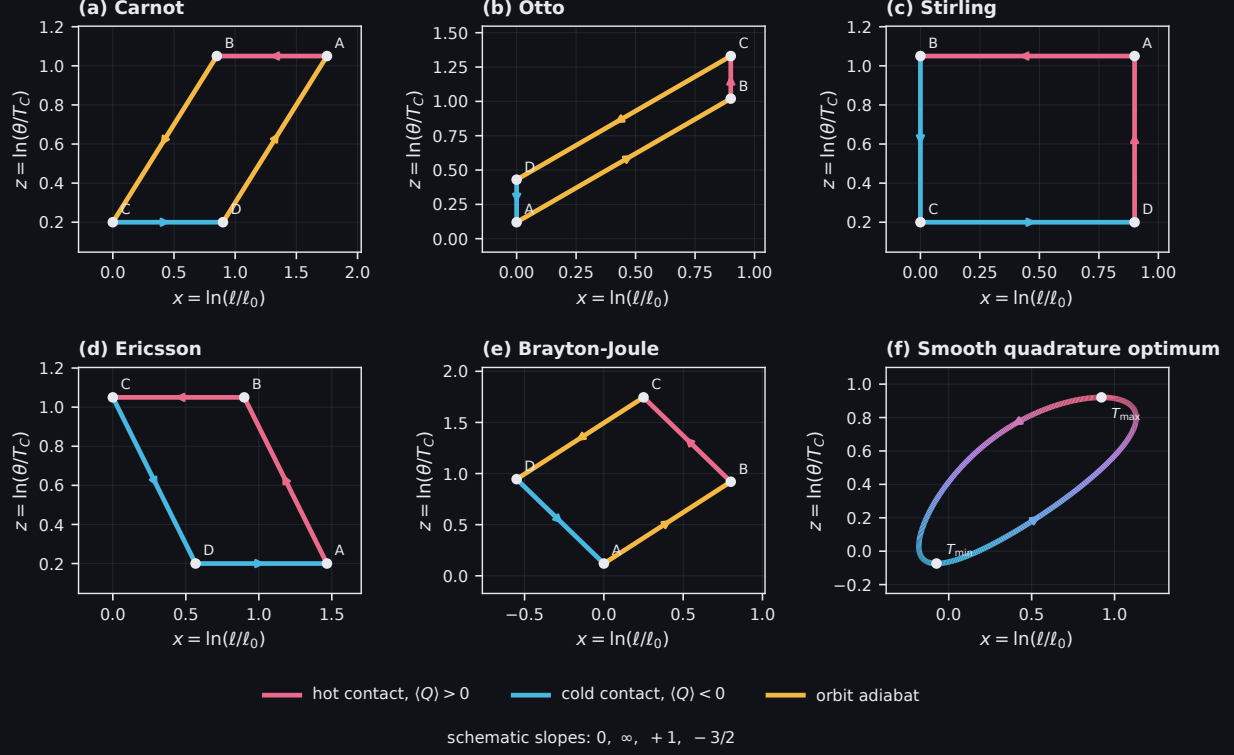


FIG. 5. Six-member constraint atlas in the coordinates $x = \ln(\ell/\ell_0)$ and $z = \ln(\theta/T_C)$. The independently scaled schematic panels use $\alpha/N = 3/2$ for the physical-gas cycles. Isotherms are horizontal, isochores vertical, isolated orbit adiabats have unit slope, and ensemble-mean isobars have slope $-\alpha/N$. Colors identify bath-contact branches and the sign of their mean heat. In panel (f), the continuously varying color records the signed mismatch $(T - \theta)/\theta$: blue rejects heat, purple marks instantaneous thermal balance, and pink absorbs heat. Regenerative bookkeeping is treated separately in the worked Stirling and Ericsson examples.

adiabatic times enter the cycle denominator, and heat leaks destroy exact endoreversibility. The Curzon–Ahlborn result below belongs to the singular four-stroke limit of this constraint family.

A. Worked example: a finite-time stochastic Otto cycle

The Otto architecture follows from a different, explicit control restriction: heat exchange is allowed only at fixed λ , whereas changes of $\ell = \sqrt{\lambda}$ are isolated and orbit adiabatic. Let $\rho = \ell_H/\ell_C > 1$ be the compression ratio and $\chi_j = \exp(-\Gamma_j \tau_j)$ for the hot and cold contacts.

At the four corners A, B, C, D of the periodic thermal manifold,

$$\begin{aligned}\theta_B &= \rho\theta_A, & \theta_D &= \theta_C/\rho, \\ \theta_C &= T_H + (\theta_B - T_H)\chi_H, & \theta_A &= T_C + (\theta_D - T_C)\chi_C.\end{aligned}\quad (112)$$

Solving the periodicity condition gives

$$\begin{aligned}\theta_A &= \frac{T_C(1 - \chi_C) + (T_H/\rho)\chi_C(1 - \chi_H)}{1 - \chi_H\chi_C}, \\ \theta_C &= \frac{T_H(1 - \chi_H) + \rho T_C\chi_H(1 - \chi_C)}{1 - \chi_H\chi_C}.\end{aligned}\quad (113)$$

The mean heats, output work, and efficiency are therefore

$$\begin{aligned}Q_H &= \alpha k_B \frac{(1 - \chi_H)(1 - \chi_C)}{1 - \chi_H\chi_C} (T_H - \rho T_C), \\ Q_C &= -\frac{Q_H}{\rho}, \quad W_{\text{out}} = \left(1 - \frac{1}{\rho}\right) Q_H, \quad \eta_O = 1 - \frac{1}{\rho}.\end{aligned}\quad (114)$$

The engine sector is $1 < \rho < T_H/T_C$, and periodicity makes the mean system entropy change vanish, so

$$\langle \Sigma_{\text{cyc}} \rangle = \frac{Q_H}{k_B} \left(\frac{1}{\rho T_C} - \frac{1}{T_H} \right) > 0. \quad (115)$$

Neglecting connector durations,

$$\mathcal{P}_O = \frac{\alpha k_B}{\tau_H + \tau_C} \left(1 - \frac{1}{\rho}\right) (T_H - \rho T_C) \frac{(1 - \chi_H)(1 - \chi_C)}{1 - \chi_H\chi_C}. \quad (116)$$

At fixed contact times and rates the last factor is independent of ρ ; maximization gives

$$\rho^* = \sqrt{\frac{T_H}{T_C}}, \quad \eta_O(\rho^*) = 1 - \sqrt{\frac{T_C}{T_H}}. \quad (117)$$

This recovers the Curzon–Ahlborn value previously obtained for classical and endoreversible harmonic Otto engines [64–66]. Here it extends across the constant- C_V homogeneous family under the stated Otto constraint; it is not a topology-independent theorem. Finite-time Brownian Otto engines and their work cumulants have also been analyzed under different, control-dependent relaxation laws [67, 75–77].

This cycle also supplies a genuine work-TUR example. In the complete cold-reset limit, the energy just before compression is an equilibrium gamma variable at T_C . The adiabats give $E_B = \rho E_A$ and $E_D = E_C/\rho$ trajectory by trajectory, hence

$$Q_H = E_C - \rho E_A, \quad W_{\text{out}} = (1 - \rho^{-1}) Q_H, \quad \Sigma_{\text{cyc}} = \left(\frac{\beta_C}{\rho} - \beta_H \right) Q_H. \quad (118)$$

The hot-contact gamma-difference law has

$$L_O = (1 - \chi_H) \frac{(T_H - \rho T_C)^2}{T_H \rho T_C}, \quad \frac{\text{Var } W_{\text{out}}}{\langle W_{\text{out}} \rangle^2} = \frac{\text{Var } Q_H}{\langle Q_H \rangle^2} = \frac{2}{\langle \Sigma_{\text{cyc}} \rangle} + \frac{1}{\alpha}. \quad (119)$$

Write $\overline{Q}_H := \langle Q_H \rangle$. Eliminating the mean entropy production between Eqs. (115) and (119) gives the directly measurable cycle law

$$\frac{\text{Var } W_{\text{out}}}{\langle W_{\text{out}} \rangle^2} = \frac{\text{Var } Q_H}{\overline{Q}_H^2} = \frac{2k_B \rho T_C T_H}{\overline{Q}_H (T_H - \rho T_C)} + \frac{1}{\alpha}. \quad (120)$$

At the compression optimum $\rho^* = \sqrt{T_H/T_C}$, the explicit compression ratio disappears:

$$\left. \frac{\text{Var } Q_H}{\overline{Q}_H^2} \right|_{\rho^*} = \frac{2k_B T_H}{\overline{Q}_H (\sqrt{T_H/T_C} - 1)} + \frac{1}{\alpha}. \quad (121)$$

Equivalently, the work affinity becomes the inverse geometric-mean temperature,

$$\Sigma_{\text{cyc}}|_{\rho^*} = \frac{W_{\text{out}}}{k_B \sqrt{T_H T_C}}, \quad \left. \frac{\text{Var } W_{\text{out}}}{\langle W_{\text{out}} \rangle^2} \right|_{\rho^*} = \frac{2k_B \sqrt{T_H T_C}}{\langle W_{\text{out}} \rangle} + \frac{1}{\alpha}. \quad (122)$$

Model-specific work distributions and cumulants are known for other stochastic Otto engines [67]; the special simplification here is the collapse of the relative variance onto the mean current, the two bath temperatures, and the thermodynamic size α . Equations (120)–(122) contain no explicit duration because the hot-contact dependence is carried by $\overline{Q}_H = \alpha k_B (1 - \chi_H)(T_H - \rho T_C)$; time has been eliminated in favor of the measured mean heat, not removed physically.

A passive complete cold reset means $\chi_C = 0$, reached by ordinary exponential relaxation only as $\tau_C \rightarrow \infty$. The full-cycle entropy-production identity above is therefore a reset or renewal limit unless an active preparation and its time and cost are included. At finite cold-contact time the trajectory identity $W_{\text{out}} = (1 - \rho^{-1})Q_H$ survives, as does Eq. (83) for the hot-stroke exchange entropy $\Sigma_{x,H} = [1/(k_B \rho \theta_A) - 1/(k_B T_H)]Q_H$; what is lost is its identification with the total cycle entropy production. Finite-time nonadiabatic strokes and heat leaks additionally destroy the heat–work tight coupling.

IX. FINITE-TIME ENGINE CYCLES ON THE GAMMA MANIFOLD

A. Endoreversible Carnot cycle and Curzon–Ahlborn performance

A one-bath cycle consumes nonnegative mean work. Engine operation requires at least two temperatures or another nonequilibrium resource. Impose one hot contact, one cold contact,

and two orbit-adiabatic connectors, and apply the variational construction of Sec. VIII to each contact. The singular arcs select a hot expansion and a cold compression at constant virtual temperatures. This recovers the endoreversible construction of Chen *et al.* [25] from work-optimal strokes under that cycle architecture. The coordinate $\ell = \sqrt{\lambda} = k^{2/(q+2)}$ gives $\delta W = E d \ln \ell$. If a residual bath remains coupled during adiabat j , the formulas below take the singular hierarchy

$$\tau_{\text{fast}} \ll \tau_{\text{ad}}^{(j)} \ll \Gamma_j^{-1}, \quad \sum_j \tau_{\text{ad}}^{(j)} \ll \tau_H + \tau_C. \quad (123)$$

The upper thermal bound is unnecessary if the bath is disconnected during the adiabats. Each stroke still contains many Hamiltonian periods, while their total duration is omitted from the displayed power denominator. If $r > 1$ is the ratio traversed by ℓ on either isotherm, then

$$\theta_H = \frac{\tau_H \Gamma_H}{\tau_H \Gamma_H + \ln r} T_H, \quad \theta_C = \frac{\tau_C \Gamma_C}{\tau_C \Gamma_C - \ln r} T_C. \quad (124)$$

The cold stroke requires $\tau_C \Gamma_C > \ln r$. The heats are

$$Q_H = \alpha k_B \Gamma_H (T_H - \theta_H) \tau_H, \quad Q_C = \alpha k_B \Gamma_C (T_C - \theta_C) \tau_C. \quad (125)$$

This realizes Newton cooling. We define positive output power by $\mathcal{P} = -\langle W_{\text{cyc}} \rangle / (\tau_H + \tau_C)$ and efficiency by $\eta = -\langle W_{\text{cyc}} \rangle / Q_H$. The engine sector is

$$T_C < \theta_C < \theta_H < T_H, \quad (126)$$

which ensures $Q_H > 0$, $Q_C < 0$, and $\mathcal{P} > 0$. Adiabatic matching gives endoreversibility, $Q_H/\theta_H + Q_C/\theta_C = 0$. The power and efficiency are

$$\begin{aligned} \mathcal{P} &= \frac{\alpha k_B \ln r}{\tau_H + \tau_C} \left[\frac{\tau_H \Gamma_H T_H}{\tau_H \Gamma_H + \ln r} - \frac{\tau_C \Gamma_C T_C}{\tau_C \Gamma_C - \ln r} \right], \\ \eta &= 1 - \frac{1 + (\tau_H \Gamma_H)^{-1} \ln r}{1 - (\tau_C \Gamma_C)^{-1} \ln r} \frac{T_C}{T_H}. \end{aligned} \quad (127)$$

At fixed r , T_H , T_C , Γ_H , and Γ_C , optimization over the two contact durations yields

$$\begin{aligned} \tau_H^* &= \frac{\ln r (\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C})}{\Gamma_H \sqrt{\Gamma_C} (\sqrt{T_H} - \sqrt{T_C})}, \\ \tau_C^* &= \frac{\ln r (\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C})}{\Gamma_C \sqrt{\Gamma_H} (\sqrt{T_H} - \sqrt{T_C})}, \\ \mathcal{P}_{\text{max}} &= \alpha k_B \Gamma_C \Gamma_H \frac{(\sqrt{T_H} - \sqrt{T_C})^2}{(\sqrt{\Gamma_H} + \sqrt{\Gamma_C})^2}, \\ \eta_{\text{EMP}} &= 1 - \sqrt{\frac{T_C}{T_H}}. \end{aligned} \quad (128)$$

The last result is the Curzon–Ahlborn efficiency [78]. It follows here under constant bath temperatures, gamma-manifold evolution, constant- θ contacts with equal logarithmic excursion, negligible-time zero-heat connectors, and optimization over contact times only. Generic low-dissipation engines have dissipation-asymmetry-dependent efficiencies [79]; stochastic-engine optimization under other control assumptions gives other efficiencies [57, 80]. With finite adiabatic duration, the denominator is $\tau_H + \tau_C + \tau_{\text{ad,tot}}$ and the durations must be reoptimized. The limit $r \rightarrow 1$ is nonuniform because the displayed optimal isothermal times are proportional to $\ln r$. The roles of the class parameters remain distinct: α scales maximum power, the Γ values determine optimal time allocation, and neither changes the efficiency at maximum power within this singular cycle. Brownian Carnot experiments provide a related finite-time setting [63].

B. Finite-time Stirling cycle

Constraint

Let $\mathcal{R} = \ell_+/\ell_- > 1$ and $A = \ln \mathcal{R}$. The Stirling architecture admits mechanical driving only on a hot constant- θ_H expansion and a cold constant- θ_C compression; bath switching occurs on two fixed- ℓ isochores. This is the stochastic Stirling constraint used in Brownian-engine theory and experiments [69, 70, 81]. The two isothermal durations are fixed by closure:

$$\tau_H^T = \frac{\theta_H A}{\Gamma_H(T_H - \theta_H)}, \quad \tau_C^T = \frac{\theta_C A}{\Gamma_C(\theta_C - T_C)}. \quad (129)$$

Periodic solution and thermodynamics

Write $\chi_\downarrow = \exp(-\Gamma_C \tau_\downarrow)$ and $\chi_\uparrow = \exp(-\Gamma_H \tau_\uparrow)$ for the cooling and heating isochores. Periodicity gives

$$\begin{aligned} \theta_H &= \frac{(1 - \chi_\uparrow)T_H + \chi_\uparrow(1 - \chi_\downarrow)T_C}{1 - \chi_\uparrow\chi_\downarrow}, \\ \theta_C &= \frac{(1 - \chi_\downarrow)T_C + \chi_\downarrow(1 - \chi_\uparrow)T_H}{1 - \chi_\uparrow\chi_\downarrow}, \\ \Delta\theta &:= \theta_H - \theta_C = (T_H - T_C) \frac{(1 - \chi_\uparrow)(1 - \chi_\downarrow)}{1 - \chi_\uparrow\chi_\downarrow}. \end{aligned} \quad (130)$$

Without a regenerator, the exact mean cycle observables are

$$\begin{aligned}
W_{\text{out}} &= \alpha k_B \Delta \theta A, \\
Q_H^{(0)} &= \alpha k_B (\theta_H A + \Delta \theta), \\
Q_C^{(0)} &= -\alpha k_B (\theta_C A + \Delta \theta), \\
\eta_{\text{St}}^{(0)} &= \frac{\Delta \theta A}{\theta_H A + \Delta \theta}, \\
\langle \Sigma_{\text{St}}^{(0)} \rangle &= \alpha \left[\frac{\theta_C A + \Delta \theta}{T_C} - \frac{\theta_H A + \Delta \theta}{T_H} \right].
\end{aligned} \tag{131}$$

A perfect-regenerator bookkeeping ideal stores and returns $\alpha k_B \Delta \theta$ on the two isochores. Its external heats and efficiency become

$$Q_H^{\text{reg}} = \alpha k_B \theta_H A, \quad Q_C^{\text{reg}} = -\alpha k_B \theta_C A, \quad \eta_{\text{St}}^{\text{reg}} = 1 - \frac{\theta_C}{\theta_H}. \tag{132}$$

Ideal regeneration is an additional component, familiar from endoreversible Stirling optimization [82]. A finite-time regenerator must have its own stochastic state and entropy production; deleting the isochoric bath heats while retaining the fixed-bath relaxation law would mix two different models.

Optimization and finding

At fixed A and fixed total isochoric time $\tau_0 = \tau_{\uparrow} + \tau_{\downarrow}$, maximizing work is equivalent to maximizing $\Delta \theta$. The allocation obeys

$$\Gamma_H \chi_{\uparrow} (1 - \chi_{\downarrow})^2 = \Gamma_C \chi_{\downarrow} (1 - \chi_{\uparrow})^2. \tag{133}$$

For equal rates this yields the closed result

$$\tau_{\uparrow}^* = \tau_{\downarrow}^* = \frac{\tau_0}{2}, \quad W_{\text{out}}^{\text{max}}(\tau_0) = \alpha k_B (T_H - T_C) A \tanh \left(\frac{\Gamma \tau_0}{4} \right). \tag{134}$$

Retaining all four stroke times creates an interior power optimum. Under equal rates and equal isochoric times, put $s = \Gamma \tau_0 / 2$ and $b = (T_H + T_C) / (T_H - T_C)$. Then

$$\mathcal{P}_{\text{St}}(s) = \alpha k_B \Gamma (T_H - T_C) A \frac{\tanh(s/2)}{2s + bA(e^s + 1)}, \tag{135}$$

and its unique positive stationary point satisfies

$$2s_* + bA(e^{s_*} + 1) = (2 + bAe^{s_*}) \sinh s_*. \tag{136}$$

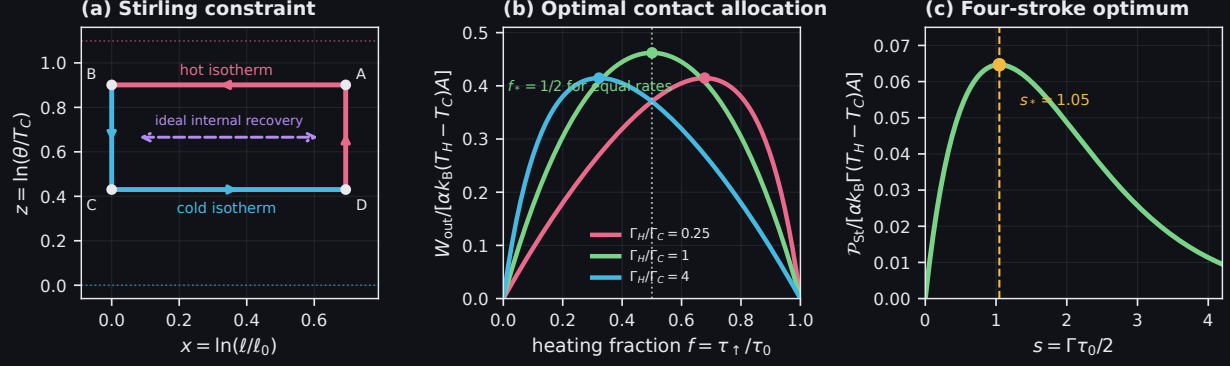


FIG. 6. Worked finite-time Stirling example for $T_H/T_C = 3$ and $A = \ln 2$. (a) The constrained cycle consists of two constant- θ work strokes and two fixed- ℓ thermal contacts; the separate dashed double arrow denotes ideal internal heat recovery, not a bath stroke. (b) At fixed total isochoric time, the optimal allocation shifts toward the slower contact and becomes symmetric for equal rates. (c) Including all four stroke times produces the interior power optimum of Eq. (136), shown for equal rates and $b = 2$. Panels (b) and (c) use the finite fixed-bath, nonregenerative dynamics.

This is a genuine four-stroke finite-time optimum. If the corner temperatures are held fixed instead, power increases monotonically with A toward a plateau, so a volume bound or control cost selects the physical expansion [83].

Fluctuation status

Every Stirling stroke has constant CIR coefficients. The joint finite-cycle transform of work and external heat is therefore an exact product of four fractional-linear Riccati maps. Cross-stroke correlations remain in that product, so the two isochoric endpoint laws cannot be multiplied as independent gamma differences. Because work contains two time-integrated contributions, the scalar Otto reset identity has no Stirling analogue without an additional tight-coupling or renewal condition.

C. Mean-isobar primitive for a physical homogeneous gas

For Eq. (109), a prescribed mean pressure imposes $V = Nk_B\theta/p$. The gamma scale then follows

$$\dot{\theta} = \frac{\Gamma}{\gamma_{\text{ad}}}(T - \theta), \quad \theta_f = T + (\theta_i - T)e^{-\Gamma\tau/\gamma_{\text{ad}}}, \quad \frac{V_f}{V_i} = \frac{\theta_f}{\theta_i}. \quad (137)$$

The exact mean stroke thermodynamics is

$$\begin{aligned} \langle W \rangle_p &= -Nk_B(\theta_f - \theta_i), \\ \langle Q \rangle_p &= (\alpha + N)k_B(\theta_f - \theta_i), \\ \langle \Sigma \rangle_p &= (\alpha + N) \left[\ln \frac{\theta_f}{\theta_i} - \frac{\theta_f - \theta_i}{T} \right]. \end{aligned} \quad (138)$$

This constraint fixes the mean pressure $\langle p \rangle$; the trajectory-level generalized pressure $p(E, V)$ continues to fluctuate. The work $W = \int a(t)E_t dt$ continues to fluctuate. Microscopic cycle definitions based on generalized pressure are discussed in Refs. [60, 61].

D. Finite-time Ericsson cycle

Constraint

The Ericsson architecture alternates a cold constant- θ_c compression, a high-pressure hot mean-isobar, a hot constant- θ_h expansion, and a low-pressure cold mean-isobar. Let $\Pi = p_H/p_L > 1$. This imposed control class has been studied in endoreversible and stochastic settings [71, 72, 84].

Periodic solution and thermodynamics

For $R_j = \exp(-\Gamma_j\tau_{p,j}/\gamma_{\text{ad}})$ and $\mathcal{D} = 1 - R_H R_C$, the two isobars give

$$\begin{aligned} \theta_c &= \frac{(1 - R_C)T_C + R_C(1 - R_H)T_H}{\mathcal{D}}, \\ \theta_h &= \frac{(1 - R_H)T_H + R_H(1 - R_C)T_C}{\mathcal{D}}, \\ \Delta\theta := \theta_h - \theta_c &= \frac{(1 - R_H)(1 - R_C)}{\mathcal{D}}(T_H - T_C). \end{aligned} \quad (139)$$

The constant-temperature stroke durations are fixed by closure:

$$\tau_{T,H} = \frac{(N/\alpha) \ln \Pi}{\Gamma_H(T_H/\theta_h - 1)}, \quad \tau_{T,C} = \frac{(N/\alpha) \ln \Pi}{\Gamma_C(1 - T_C/\theta_c)}. \quad (140)$$

Without regeneration, the mean cycle observables are

$$\begin{aligned}
W_{\text{out}} &= Nk_B\Delta\theta \ln \Pi, \\
Q_H &= Nk_B\theta_h \ln \Pi + (\alpha + N)k_B\Delta\theta, \\
Q_C &= -Nk_B\theta_c \ln \Pi - (\alpha + N)k_B\Delta\theta, \\
\eta_E^{(0)} &= \frac{N\Delta\theta \ln \Pi}{N\theta_h \ln \Pi + (\alpha + N)\Delta\theta}, \\
\langle \Sigma_E^{(0)} \rangle &= N \ln \Pi \left(\frac{\theta_c}{T_C} - \frac{\theta_h}{T_H} \right) + (\alpha + N)\Delta\theta \left(\frac{1}{T_C} - \frac{1}{T_H} \right). \tag{141}
\end{aligned}$$

Perfect regenerative bookkeeping removes the two isobaric heats from the external budget and gives $\eta_E^{\text{reg}} = 1 - \theta_c/\theta_h$. As in the Stirling case, this familiar limit is the regenerative benchmark. Equation (139) gives the finite fixed-bath model.

Optimization and finding

At fixed isobaric contact times, write

$$\mathcal{A}_E = Nk_B\Delta\theta, \quad \mathcal{B}_E = \frac{N/\alpha}{\Gamma_H(T_H/\theta_h - 1)} + \frac{N/\alpha}{\Gamma_C(1 - T_C/\theta_c)}, \quad \tau_p = \tau_{p,H} + \tau_{p,C}. \tag{142}$$

The pressure-ratio problem is then

$$\mathcal{P}_E(\Pi) = \frac{\mathcal{A}_E \ln \Pi}{\tau_p + \mathcal{B}_E \ln \Pi}, \quad \frac{\partial \mathcal{P}_E}{\partial \ln \Pi} > 0, \quad \lim_{\Pi \rightarrow \infty} \mathcal{P}_E = \frac{\mathcal{A}_E}{\mathcal{B}_E}. \tag{143}$$

Thus the unconstrained pressure ratio has no finite maximizer: a pressure or volume bound, pressure-dependent relaxation, wall-speed limit, or superlinear actuation cost supplies the missing scale. At fixed total isobaric time, $\Delta\theta$ is maximized by Eq. (149) below; equal relaxation rates allocate equal times. Optimization of full power also differentiates the θ_c, θ_h dependence of $\mathcal{B}_E$ and is generally implicit.

Fluctuation status

The two constant- θ strokes have elementary constant-coefficient Riccati transforms. On each isobar, $a(t) = -(N/\alpha)\dot{\theta}/\theta$ is time dependent, so its exact transform is defined by the corresponding Riccati propagator. This gives an exact joint MGF representation, while a cycle-level scalar TUR equality requires trajectory tight coupling.

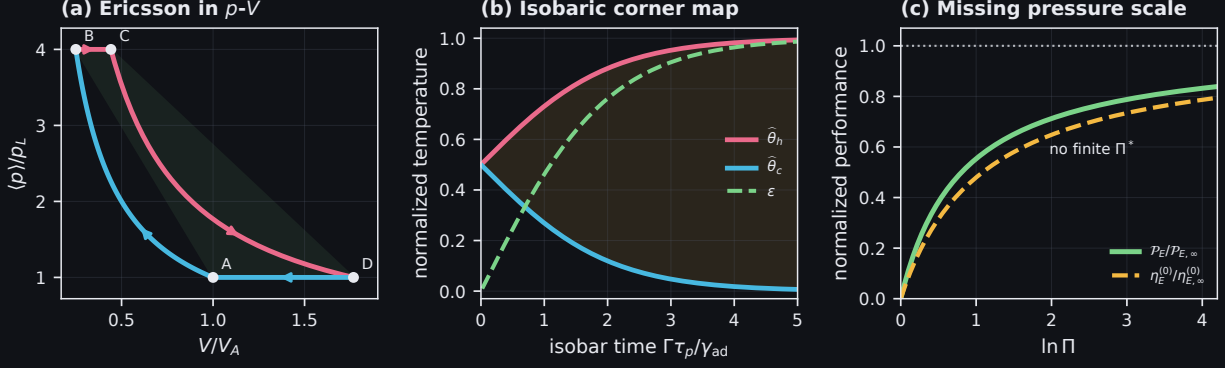


FIG. 7. Worked finite-time Ericsson example for $T_H/T_C = 4$, $R_H = R_C = e^{-1}$, and $\alpha/N = 3/2$. (a) The physical-gas cycle in ensemble-mean pressure–volume coordinates. (b) The exact periodic corner temperatures and their normalized separation as the isobaric contact time is increased. (c) Both normalized power and nonregenerative efficiency approach plateaus with $\ln \Pi$; the unconstrained model therefore selects no finite pressure ratio. The plotted thermodynamic loop is exact at the gamma-manifold mean level; trajectory work on each isobar remains a Riccati path functional.

E. Finite-time Brayton–Joule cycle

Constraint

The Brayton–Joule architecture alternates isolated orbit-adiabatic compression, a high-pressure hot mean-isobar, isolated orbit-adiabatic expansion, and a low-pressure cold mean-isobar [60, 73, 85]. Define

$$\Pi = \frac{p_H}{p_L}, \quad r = \Pi^{(\gamma_{\text{ad}}-1)/\gamma_{\text{ad}}} = \Pi^{N/(\alpha+N)}, \quad R_j = e^{-\Gamma_j \tau_j / \gamma_{\text{ad}}}. \quad (144)$$

Periodic solution and thermodynamics

Put $c_j = 1 - R_j$, $\mathcal{D} = 1 - R_H R_C$, and $\varepsilon = c_H c_C / \mathcal{D}$. The exact corner map is

$$\begin{aligned} \theta_B &= r \theta_A, & \theta_C &= c_H T_H + R_H r \theta_A, \\ \theta_D &= \theta_C / r, & \theta_A &= \frac{c_C T_C + (R_C c_H / r) T_H}{\mathcal{D}}. \end{aligned} \quad (145)$$

It implies

$$\begin{aligned}
Q_H &= (\alpha + N)k_B\varepsilon(T_H - rT_C), \\
Q_C &= -\frac{Q_H}{r}, \quad W_{\text{out}} = \left(1 - \frac{1}{r}\right)Q_H, \quad \eta_B = 1 - \frac{1}{r}, \\
\langle \Sigma_B \rangle &= (\alpha + N)\varepsilon \frac{(T_H - rT_C)^2}{rT_C T_H}.
\end{aligned} \tag{146}$$

The engine domain is $1 < r < T_H/T_C$. These mean and corner relations are exact on the gamma manifold; the fluctuating isobaric work remains a temporal functional.

Optimization and finding

When the contact factors and cycle duration are independent of Π , power and work share the pressure-ratio optimum

$$r^* = \sqrt{\frac{T_H}{T_C}}, \quad \Pi^* = \left(\frac{T_H}{T_C}\right)^{(\alpha+N)/(2N)}, \quad \eta_B^* = 1 - \sqrt{\frac{T_C}{T_H}}. \tag{147}$$

This recovers the Curzon–Ahlborn value in a Brayton constraint class under fixed contact factors and pressure-independent cycle duration [60, 73, 85], while the optimal physical pressure ratio retains the universality parameter α/N . At the optimum,

$$W_{\text{out}}^* = (\alpha + N)k_B\varepsilon(\sqrt{T_H} - \sqrt{T_C})^2, \quad \langle \Sigma_B \rangle^* = \frac{W_{\text{out}}^*}{k_B\sqrt{T_H T_C}}. \tag{148}$$

The second relation is a mean identity. Work on the mean-isobar is not trajectory-wise proportional to the endpoint heat, so it does not by itself imply the Otto work-TUR equality.

For fixed $\tau_H + \tau_C$, maximizing the contact factor gives, with $x = \Gamma_H\tau_H/\gamma_{\text{ad}}$ and $y = \Gamma_C\tau_C/\gamma_{\text{ad}}$,

$$\frac{\Gamma_H}{\gamma_{\text{ad}}} \left[\coth \frac{x}{2} - \coth \frac{x+y}{2} \right] = \frac{\Gamma_C}{\gamma_{\text{ad}}} \left[\coth \frac{y}{2} - \coth \frac{x+y}{2} \right]. \tag{149}$$

Equal rates give $\tau_H^* = \tau_C^*$. Finite connector times that grow with $\ln \Pi$, pressure-dependent rates, or wall-speed bounds shift Eq. (147).

Fluctuation status

The four corner marginals and the displayed mean thermodynamics are exact on the gamma manifold. Isobaric work, however, is a time integral with a time-dependent coefficient and is determined by a Riccati path transform. Without an additional trajectory-level tight-coupling condition, the mean identity in Eq. (148) is not a scalar work-TUR equality.

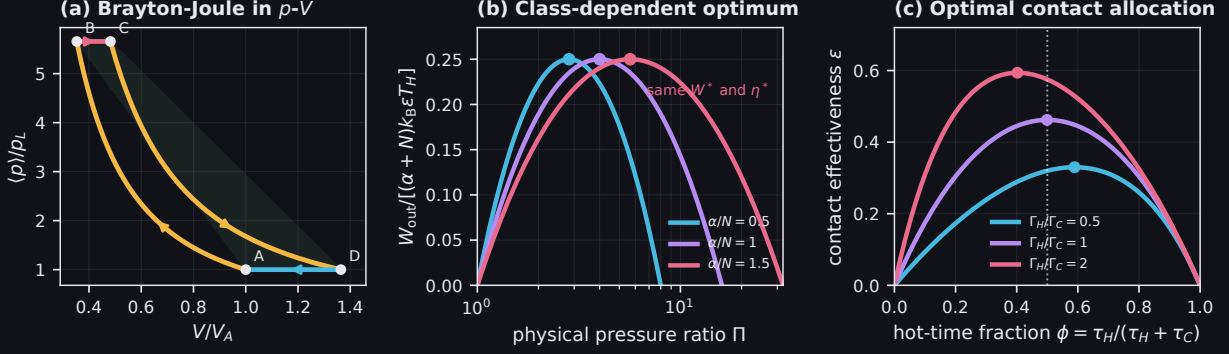


FIG. 8. Worked finite-time Brayton-Joule example for $T_H/T_C = 4$. (a) The ensemble-mean pressure-volume loop at the optimum for $\alpha/N = 3/2$ and $R_H = R_C = e^{-1}$. (b) The maximum dimensionless work and efficiency occur at the same thermal ratio for every homogeneous class, while the physical optimum Π^* shifts with α/N . (c) At fixed total contact time, unequal relaxation rates shift the optimal hot-time fraction according to Eq. (149). The pressure-ratio result assumes fixed contact factors and a pressure-independent cycle duration.

F. Globally optimal smooth quadrature cycle

Chen and Quan’s maximum-power problem under stiffness and bath-switching constraints selects a different optimal topology with an additional isochoric relaxation [26]. Accordingly, “global” below means global only among the closed gamma-manifold loops satisfying Eq. (152), with the positivity domain in Eq. (157); actuator, stiffness, temperature, and switching bounds are not part of this theorem.

The first five atlas entries prescribe familiar stroke topologies. We now pose an exact continuous-control exercise on the same gamma manifold: find the closed loop, including both the confinement and bath-temperature waveforms, that delivers the greatest mean work under experimentally legible quadratic budgets. This formulation is related to finite-time Brownian control and periodic heat-engine optimization [57, 74, 75, 80]; the result below is the closed-form solution of the present homogeneous energy-space problem.

Constraint

Let

$$x = \ln(\ell/\ell_0), \quad z = \ln(\theta/T_C), \quad s = z - x, \quad u = \frac{T}{\theta} - 1. \quad (150)$$

The gamma-manifold equation and the entropy differential in Eq. (108) become

$$\dot{x} = \dot{z} - \Gamma_q u, \quad \dot{s} = \Gamma_q u. \quad (151)$$

For precision, the admissible class consists of positive $\theta \in H_{\text{per}}^1(0, \tau)$ bounded away from zero and $u \in L_{\text{per}}^2(0, \tau)$ with $\langle u \rangle_\tau = 0$ and $1 + u > 0$ almost everywhere. The bath control is $T = \theta(1 + u)$, and the confinement coordinate $x \in H_{\text{per}}^1(0, \tau)$ is reconstructed, up to an additive constant, from Eq. (151). Over a fixed period $\tau = 2\pi/\omega$, denote $\langle g \rangle_\tau = \tau^{-1} \int_0^\tau g(t) dt$. The closed-loop conditions $x(t + \tau) = x(t)$ and $\theta(t + \tau) = \theta(t)$ imply $\langle u \rangle_\tau = 0$ through Eq. (151). We then impose

$$\langle \theta \rangle_\tau = \bar{\theta}, \quad \langle \dot{\theta}^2 \rangle_\tau = R_\theta^2, \quad \langle u^2 \rangle_\tau = M^2. \quad (152)$$

Here $\alpha k_B R_\theta$ is the RMS rate of change of mean energy, whereas M is the RMS fractional mismatch between the bath and the state. The controls $\theta(t)$ and $u(t)$ are optimized; the physical controls $T(t)$ and $\ell(t)$ are reconstructed from Eq. (151).

The positive output power of any periodic admissible loop is

$$\mathcal{P}_{\text{sm}} = -\frac{\langle W \rangle_{\text{cyc}}}{\tau} = \alpha k_B \Gamma_q \langle \theta u \rangle_\tau = \alpha k_B \Gamma_q \langle (\theta - \bar{\theta}) u \rangle_\tau. \quad (153)$$

Cauchy–Schwarz and the periodic Poincaré–Wirtinger inequality [86] give the global bound

$$\mathcal{P}_{\text{sm}} \leq \alpha k_B \Gamma_q M \sqrt{\langle (\theta - \bar{\theta})^2 \rangle_\tau} \leq \frac{\alpha k_B \Gamma_q R_\theta M}{\omega}. \quad (154)$$

Optimal solution

Introduce the dimensionless amplitudes

$$\epsilon = \frac{\sqrt{2} R_\theta}{\omega \bar{\theta}}, \quad \delta = \sqrt{2} M, \quad h = \frac{\Gamma_q \delta}{\omega}. \quad (155)$$

Equality in both steps of Eq. (154) fixes a single Fourier mode and places the thermal mismatch in phase with it:

$$\begin{aligned} \theta_*(t) &= \bar{\theta}[1 + \epsilon \cos(\omega t)], & u_*(t) &= \delta \cos(\omega t), \\ T_*(t) &= \bar{\theta}[1 + \epsilon \cos(\omega t)][1 + \delta \cos(\omega t)], \\ z_*(t) &= \ln \frac{\bar{\theta}}{T_C} + \ln[1 + \epsilon \cos(\omega t)], & x_*(t) &= x_0 + z_*(t) - h \sin(\omega t). \end{aligned} \quad (156)$$

Thus the quadrature component of x_* is derived from the optimization; it was not prescribed as an orbit. The regular engine domain

$$0 < \epsilon < 1, \quad 0 < \delta < 1 \quad \Longleftrightarrow \quad R_\theta < \frac{\omega \bar{\theta}}{\sqrt{2}}, \quad M < \frac{1}{\sqrt{2}} \quad (157)$$

keeps both θ_* and T_* positive. Within the admissible periodic space just stated, the equality conditions of Cauchy–Schwarz and Poincaré–Wirtinger make time translation and the additive constant x_0 the only degeneracies of the maximizing loop. The maximal power and work per cycle are

$$\mathcal{P}_{\text{sm}}^{\text{max}} = \frac{\alpha k_{\text{B}} \Gamma_q R_\theta M}{\omega} = \frac{\alpha k_{\text{B}} \Gamma_q \bar{\theta} \epsilon \delta}{2}, \quad W_{\text{out}}^{\text{max}} = \pi \alpha k_{\text{B}} \bar{\theta} h \epsilon. \quad (158)$$

Thermodynamics and finding

Heat absorption occurs on the half-cycle $u_* > 0$ and rejection on $u_* < 0$. Direct integration gives

$$\begin{aligned} Q_+ &= \alpha k_{\text{B}} \bar{\theta} h \left(2 + \frac{\pi \epsilon}{2} \right), & Q_- &= \alpha k_{\text{B}} \bar{\theta} h \left(-2 + \frac{\pi \epsilon}{2} \right), \\ Q_+ + Q_- &= W_{\text{out}}^{\text{max}}, & \eta_{\text{sm}} &= \frac{W_{\text{out}}^{\text{max}}}{Q_+} = \frac{2\pi \epsilon}{4 + \pi \epsilon}. \end{aligned} \quad (159)$$

This is a continuously temperature-programmed reservoir, so Q_+ and Q_- are positive- and negative-heat sectors of that programmed contact. The displayed efficiency excludes the energetic cost of operating that thermostat and carries no two-reservoir Carnot interpretation.

The exact mean dimensionless entropy production is

$$\langle \Sigma_{\text{sm}} \rangle = \alpha \Gamma_q \int_0^\tau \frac{u_*^2(t)}{1 + u_*(t)} dt = \frac{2\pi \alpha \Gamma_q}{\omega} \left[\frac{1}{\sqrt{1 - \delta^2}} - 1 \right] = \frac{2\pi \alpha \Gamma_q}{\omega} \left[\frac{1}{\sqrt{1 - 2M^2}} - 1 \right]. \quad (160)$$

The RMS mismatch in Eq. (152) is therefore a quadratic control budget, not fixed entropy production. Replacing it by a fixed value of the nonlinear functional in Eq. (160) produces a nonlinear optimizer; Eq. (156) is its controlled weak-mismatch limit.

An outer optimization becomes closed when the experiment fixes the extrema T_{max} and T_{min} of the programmed bath. Write $\epsilon = \tanh a$ and $\delta = \tanh b$. Then

$$a + b = \frac{1}{2} \ln \frac{T_{\text{max}}}{T_{\text{min}}}, \quad \mathcal{P}_{\text{sm}}^{\text{max}} = \frac{\alpha k_{\text{B}} \Gamma_q \sqrt{T_{\text{max}} T_{\text{min}}}}{2} \sinh a \sinh b. \quad (161)$$

At fixed span the rate and mismatch budgets split equally:

$$\begin{aligned}
a^* &= b^*, & \epsilon^* &= \delta^* = \frac{\sqrt{T_{\max}} - \sqrt{T_{\min}}}{\sqrt{T_{\max}} + \sqrt{T_{\min}}}, \\
\mathcal{P}_{\text{span}}^{\max} &= \frac{\alpha k_B \Gamma_q}{8} \left(\sqrt{T_{\max}} - \sqrt{T_{\min}} \right)^2, & \eta_{\text{sm}}^* &= \frac{2\pi\epsilon^*}{4 + \pi\epsilon^*}, \\
\bar{\Sigma}_{\text{sm}}^* &= \alpha \Gamma_q \left[\cosh \left(\frac{1}{4} \ln \frac{T_{\max}}{T_{\min}} \right) - 1 \right].
\end{aligned} \tag{162}$$

The experimentally useful conclusion is a balance law: with a prescribed temperature span, maximum power assigns equal logarithmic amplitude to state slew and bath–state mismatch.

Fluctuation status

The normalized energy and operational time

$$Y_t = \frac{E_t}{k_B \theta_*(t)}, \quad \mathbf{u}(t) = \Gamma_q \int_0^t [1 + u_*(r)] dr = \Gamma_q t + \frac{\Gamma_q \delta}{\omega} \sin(\omega t) \tag{163}$$

reduce the driven process exactly to

$$dY = (\alpha - Y) du + \sqrt{2Y} dB_u. \tag{164}$$

In its periodic gamma state, $Y_t \sim \text{Gamma}(\alpha, 1)$ and

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t'}) &= \alpha \exp[-|\mathbf{u}(t) - \mathbf{u}(t')|], \\
\text{Cov}(E_t, E_{t'}) &= \alpha k_B^2 \theta_*(t) \theta_*(t') \exp[-|\mathbf{u}(t) - \mathbf{u}(t')|], \\
\text{Corr}(Y_{t+n\tau}, Y_t) &= e^{-n\Gamma_q \tau}.
\end{aligned} \tag{165}$$

Consequently the cycle-work variance is the exact double quadrature

$$\text{Var } W_{\text{cyc}} = \alpha k_B^2 \int_0^\tau \int_0^\tau \theta_*(t) \theta_*(t') \dot{x}_*(t) \dot{x}_*(t') e^{-|\mathbf{u}(t) - \mathbf{u}(t')|} dt dt'. \tag{166}$$

Higher work cumulants follow from the time-dependent affine Riccati propagator. Equation (166) is exact and does not assume Gaussian work or a scalar TUR equality.

G. Cross-cycle comparison

The atlas separates three different appearances of the Curzon–Ahlborn number. Carnot obtains it by optimizing two finite-time isothermal contact durations. Otto and Brayton

obtain it by optimizing a mechanical ratio while holding their thermal contact factors fixed. Stirling has instead an interior contact-time optimum, Eq. (136); Ericsson exposes a missing-scale diagnosis, Eq. (143). The smooth cycle optimizes the entire closed waveform and saturates a global Poincaré–Wirtinger bound; a fixed temperature span then selects equal logarithmic state-slew and mismatch amplitudes. These are distinct constrained variational statements. Their quasistatic fluctuation relations belong to the general phase-volume theory of Ref. [61]; the finite-time results here retain the kinetic scales Γ_j and the corner correlations carried by the CIR propagators.

X. INDEPENDENT SUBSYSTEMS AND INTERACTION NETWORKS

A. Independent subsystems

At fixed controls, let independent subsystem energies obey

$$dE_i = \Gamma_i(\alpha_i k_B T - E_i)dt + \sqrt{2\Gamma_i k_B T E_i} dB_i. \quad (167)$$

Their equilibrium total energy has shape $\alpha_{\text{tot}} = \sum_i \alpha_i$. For arbitrary rates, its drift and quadratic variation are

$$\begin{aligned} b_{\text{tot}} &= k_B T \sum_i \Gamma_i \alpha_i - \sum_i \Gamma_i E_i, \\ d[E_{\text{tot}}]_t &= 2k_B T \sum_i \Gamma_i E_i dt. \end{aligned} \quad (168)$$

For nondegenerate independent partitions, both depend only on E_{tot} for all states if and only if all Γ_i are equal. The total then closes as

$$dE_{\text{tot}} = \Gamma(\alpha_{\text{tot}} k_B T - E_{\text{tot}})dt + \sqrt{2\Gamma k_B T E_{\text{tot}}} dB. \quad (169)$$

If the rates differ, the unresolved partition remains in Eq. (168), so E_{tot} is generally non-Markovian by itself. Under driving, scalar closure further requires common protocol drifts $a_i(t)$, or a relation making $\sum_i a_i E_i$ a function of E_{tot} .

For a common rate, the equilibrium fractions satisfy

$$(x_1, \dots, x_N), \quad x_i = \frac{E_i}{E_{\text{tot}}} \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_N). \quad (170)$$

Itô's formula gives the fraction dynamics

$$\begin{aligned} dx_i &= \frac{\Gamma k_B T}{E_{\text{tot}}} (\alpha_i - \alpha_{\text{tot}} x_i) dt \\ &\quad + \sqrt{\frac{2\Gamma k_B T}{E_{\text{tot}}}} \left[\sqrt{x_i} dB_i - x_i \sum_j \sqrt{x_j} dB_j \right]. \end{aligned} \quad (171)$$

After the random time change $d\tau = 2\Gamma k_B T dt / E_{\text{tot}}$, Eq. (171) is a Wright–Fisher diffusion with stationary law (170) [87]. This decomposition exposes information discarded by the scalar total energy even when that scalar process closes exactly.

B. Universal total-energy sector

Let $G = (g_{ij})$ be the symmetric weight matrix of a connected graph and consider the pair potential

$$U_G(\mathbf{r}; k) = \frac{k}{q} \sum_{i < j} g_{ij} |\mathbf{r}_i - \mathbf{r}_j|^q, \quad g_{ij} \geq 0. \quad (172)$$

Remove the free center-of-mass coordinate and momentum and thermostat the $D = d_s(N-1)$ internal momentum modes uniformly. Scaling all internal momenta by $E^{1/2}$ and all relative coordinates by $(E/k)^{1/q}$ gives

$$\begin{aligned} \Omega(E, k, G) &= \mathcal{C}_q(G, M) k^{-D/q} E^\alpha, \\ \alpha &= D \left(\frac{1}{2} + \frac{1}{q} \right), \quad f_{\text{eff}} = D \left(1 + \frac{2}{q} \right), \\ \Gamma_q &= \frac{2q}{q+2} \gamma. \end{aligned} \quad (173)$$

Thus (D, q, γ) determine the normalized total-energy law, $C_V = \alpha k_B$, and the relaxation rate. The matrix G and the masses remain in the phase-volume coefficient $\mathcal{C}_q$.

For any bounded homogeneous orbit, Euler's theorem for U_G and the vanishing time average of $d(\sum_a R_a P_a)/dt$ give the virial identities

$$\overline{\sum_a P_a^2 / M_a} = \frac{2q}{q+2} E, \quad \overline{\partial_k H} = \frac{2E}{(q+2)k}. \quad (174)$$

The fixed- G total energy is therefore lumpable at leading order, including integrable internal dynamics. Shape protocols that vary individual entries of G require an additional condition because their work coefficients resolve bond energies.

C. Graph-specific equilibrium thermodynamics

Let $\mathcal{X}$ denote the D -dimensional center-of-mass-free configuration space and define

$$\mathcal{Z}_q(G) = \int_{\mathcal{X}} d^D y \exp \left[-\frac{1}{q} \sum_{i < j} g_{ij} |\mathbf{y}_i - \mathbf{y}_j|^q \right]. \quad (175)$$

With a fixed phase-space unit $\mathfrak{h}$, the internal canonical partition function factorizes as

$$Z_{\text{int}}(T, k, G) = \frac{(2\pi k_B T)^{D/2} \sqrt{\det M_{\text{int}}}}{\mathfrak{h}^D} \left(\frac{k_B T}{k} \right)^{D/q} \mathcal{Z}_q(G). \quad (176)$$

All graph-specific equilibrium thermodynamics is contained in the single scalar function $\mathcal{Z}_q(G)$:

$$F_{\text{int}} = -k_B T \ln Z_{\text{int}}, \quad S_{\text{int}} = k_B (\ln Z_{\text{int}} + \alpha), \quad C_{k,G} = \alpha k_B. \quad (177)$$

Consequently, two connected graphs of the same rank, masses, q , and overall scale k obey

$$\Delta_G \langle E \rangle = 0, \quad \Delta_G F = -k_B T \ln \frac{\mathcal{Z}_q(G_1)}{\mathcal{Z}_q(G_0)}, \quad \Delta_G S = k_B \ln \frac{\mathcal{Z}_q(G_1)}{\mathcal{Z}_q(G_0)}, \quad \Delta_G C = 0. \quad (178)$$

The coupling matrix therefore changes the entropic part of the equation of state while leaving the mean internal energy and heat capacity in the same homogeneous universality class. For an edge $e = (i, j)$, its generalized force and the isothermal response matrix are

$$\begin{aligned} \mathcal{X}_e &:= \frac{\partial F_{\text{int}}}{\partial g_e} = \frac{k_B T}{q} \langle |\mathbf{y}_i - \mathbf{y}_j|^q \rangle_G, \\ \frac{\partial \mathcal{X}_e}{\partial g_f} &= -\frac{k_B T}{q^2} \text{Cov}_G(|\Delta \mathbf{y}_e|^q, |\Delta \mathbf{y}_f|^q). \end{aligned} \quad (179)$$

Uniform rescaling of all edge weights yields the graph equation of state

$$\sum_e g_e \mathcal{X}_e = \frac{D}{q} k_B T = \langle U_G \rangle. \quad (180)$$

For a shape-changing protocol whose shell averages are microcanonical, Eq. (13) remains a driven CIR process with

$$a_G(t) = \frac{D}{\alpha q} \frac{\dot{k}}{k} - \frac{1}{\alpha} \frac{d}{dt} \ln \mathcal{Z}_q(G_t). \quad (181)$$

The first term follows orbitwise from Eq. (174). The second term requires shell mixing or another identity that makes the graph-shape work depend only on total energy. When scalar closure fails, additional orbit labels or averaged observables may be required; examples include normal-mode actions in integrable harmonic networks and suitable bond-energy combinations. Determining a minimal closed set is model dependent.

D. Harmonic matrix class

The case $q = 2$ exposes the matrix dependence spectrally. Determinant-dependent entropies are established in the statistical mechanics of Gaussian molecules and coarse-grained elastic networks, and thermal harmonic-network fluctuations have been related to resistance-based graph observables [88–90]. Here these structures are integrated with the homogeneous energy-diffusion reduction. Write

$$U_K(\mathbf{r}; k) = \frac{k}{2} \mathbf{r}^\top (K \otimes I_{d_s}) \mathbf{r}, \quad K \succeq 0, \quad (182)$$

and assume that K has fixed rank r and fixed null space along a protocol. On its confined subspace $D = d_s r$, and direct Gaussian integration gives

$$\Omega(E, k, K) = \frac{(2\pi)^D}{\Gamma(D+1)} \frac{E^D \sqrt{\det M_{\text{int}}}}{k^{D/2} [\det' K]^{d_s/2}}. \quad (183)$$

Here $\det'$ denotes the product of nonzero eigenvalues. At fixed T and k ,

$$\Delta F_K = \frac{d_s k_B T}{2} \ln \frac{\det' K_f}{\det' K_i}, \quad C_{k,K} = D k_B. \quad (184)$$

For pair springs $K = L(G)$ is the weighted graph Laplacian. The matrix-tree theorem and the resistance-distance identity give [91, 92]

$$\begin{aligned} \det' L(G) &= N \tau(G), \quad \tau(G) = \sum_{\mathcal{T}} \prod_{e \in \mathcal{T}} g_e, \\ R_{ij} &= (\mathbf{e}_i - \mathbf{e}_j)^\top L^+ (\mathbf{e}_i - \mathbf{e}_j), \\ \frac{\partial F_{\text{int}}}{\partial g_{ij}} &= \frac{d_s k_B T}{2} R_{ij}, \quad \langle |\mathbf{r}_i - \mathbf{r}_j|^2 \rangle = \frac{d_s k_B T}{k} R_{ij}. \end{aligned} \quad (185)$$

Thus the graph-dependent part of the equilibrium free energy is $(d_s k_B T/2) \ln \tau(G)$ up to a graph-independent constant, while an edge perturbation probes its effective resistance.

The full isothermal response is also explicit. With the oriented incidence vector $\mathbf{b}_e = \mathbf{e}_i - \mathbf{e}_j$ and fixed Laplacian rank,

$$\frac{\partial^2 F_{\text{int}}}{\partial g_e \partial g_f} = -\frac{d_s k_B T}{2} (\mathbf{b}_e^\top L^+ \mathbf{b}_f)^2. \quad (186)$$

The negative-semidefinite Hessian is the harmonic specialization of the covariance response in Eq. (179). A second global observable is the Kirchhoff index

$$\text{Kf}(G) := \sum_{i < j} R_{ij} = N \text{tr} L^+ = N \sum_{a=2}^N \lambda_a^{-1}. \quad (187)$$

TABLE II. Exact harmonic graph observables for N vertices.

Graph	$\text{tw}(G)$	$\tau(G)$	$\text{Kf}(G)$
Path P_N	1	1	$(N^3 - N)/6$
Star S_N	1	1	$(N - 1)^2$
Cycle C_N	2	N	$(N^3 - N)/12$
Complete K_N	$N - 1$	N^{N-2}	$N - 1$

It controls the thermal size of the network through the radius of gyration,

$$\langle R_g^2 \rangle = \frac{1}{N} \sum_i \langle |\mathbf{r}_i - \mathbf{r}_{\text{cm}}|^2 \rangle = \frac{d_s k_B T}{k N^2} \text{Kf}(G). \quad (188)$$

Hence $\ln \tau(G)$ is a free-energy observable, R_{ij} is a local compliance, and $\text{Kf}(G)$ is a thermodynamic size observable. For equal particle masses m , the lowest nonzero normal-mode frequency satisfies $\omega_{\min}^2 = (k/m)\lambda_2[L(G)]$ and helps test the fast-orbit condition. For unequal masses, the relevant quantity is the smallest positive generalized eigenvalue of $L\mathbf{v} = \mu M\mathbf{v}$.

For unit-weight connected graphs these observables separate familiar graph families: Path and star have identical spanning-tree free energy while their radii of gyration scale respectively as $O(N)$ and $O(1)$. The cycle has $O(N)$ size with half the path prefactor, and the complete graph contracts as $O(N^{-1})$. A single scalar graph free energy therefore leaves measurable geometry unresolved.

There is a precise connection to treewidth through the multivariate spanning-tree polynomial $\tau_G(\mathbf{g})$. The harmonic configurational factor is proportional to $\tau_G(\mathbf{g})^{-\beta}$ with $\beta = d_s/2$. Complete monotonicity means

$$(-1)^{|\mathbf{n}|} \partial_{\mathbf{g}}^{\mathbf{n}} \tau_G(\mathbf{g})^{-\beta} \geq 0 \quad (\mathbf{g} > 0) \quad (189)$$

for every edge multi-index $\mathbf{n}$. At physical integer d_s , Eq. (189) follows for every graph from the positive Gaussian Laplace representation of the configurational integral [93]. For a finite connected simple graph with $g_e > 0$, a recent preprint proves the sufficient certificate

$$\text{tw}(G) \leq k, \quad \beta > \frac{k-1}{2} \implies \tau_G^{-\beta} \text{ is completely monotone} \quad (190)$$

[94]. For non-half-integer β , this result establishes a positive Laplace-transform representation of the analytically continued configurational factor; interpreting 2β as a physical spatial

dimension is formal. Treewidth is sufficient here and does not classify all completely monotone cases. It also does not determine $\tau(G)$, $\text{Kf}(G)$, or the stiffness spectrum: the path and star already share treewidth one and have different thermodynamic sizes. Physical graph classes are resolved by the polynomial and spectrum, with treewidth providing an additional structural and sparse-elimination certificate.

The two matrix roles should be kept distinct. In the pair model (172), only off-diagonal edge weights enter: $G = I_N$ has no pair interactions, whereas $G = J_N - I_N$ (equivalently J_N after ignoring its diagonal) is the complete graph and produces $K = L(G) = NI_N - J_N$. By contrast, K in Eq. (182) is the physical stiffness itself. Hence $K = I_N$ describes independent onsite oscillators, while $K = J_N$ confines only one collective coordinate.

Independent onsite traps have $K = I_N$, rank N , and $\det K = 1$. The literal all-ones stiffness $K = J_N = \mathbf{1}\mathbf{1}^\top$ has rank one and leaves $N - 1$ particle-space directions free. The positive-definite interpolation

$$K_\rho = (1 - \rho)I_N + \rho J_N, \quad \det K_\rho = (1 - \rho)^{N-1}[1 + (N - 1)\rho] \quad (191)$$

connects I_N to the singular all-ones limit as $\rho \rightarrow 1^-$. The vanishing determinant records the diverging configurational volume of the released modes.

All-to-all *pair* springs use the complete-graph Laplacian

$$L_{\text{complete}} = NI_N - J_N, \quad \text{rank } L_{\text{complete}} = N - 1, \quad \det' L_{\text{complete}} = N^{N-1}. \quad (192)$$

After center-of-mass removal this matrix is simply NI on the internal subspace. More general graphs share the universal gamma energy law while differing through $\tau(G)$, R_{ij} , graph entropy, and shape-work fluctuations.

For a translationally invariant pair network placed in a large container of volume $\mathcal{V}$, the center-of-mass factors restore the mechanical gas variables. If the interaction graph has c connected components, boundary corrections neglected, equilibrium gives

$$p\mathcal{V} = ck_{\text{B}}T + o(1), \quad C_V^{\text{full}} = \left[\frac{d_s N}{2} + \frac{d_s(N - c)}{q} \right] k_{\text{B}}. \quad (193)$$

Thus $G = I_N$ in the pair-weight convention gives $c = N$, the ideal-gas laws $p\mathcal{V} = Nk_{\text{B}}T$ and $C_V = d_s Nk_{\text{B}}/2$, while a complete connected network gives $c = 1$ and describes one bound molecular cluster. An extensive large- N network limit requires an explicit normalization of the edge weights. The pressure detects connected components; within each connected sector,

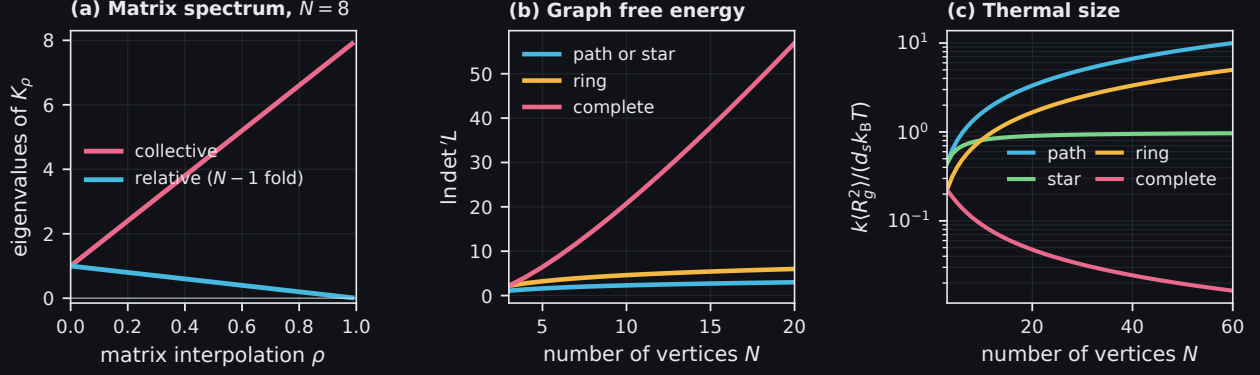


FIG. 9. Matrix-dependent thermodynamics of harmonic networks. (a) The spectrum of $K_\rho = (1 - \rho)I + \rho J$ separates collective and relative modes. (b) The free-energy factor $\ln \det' L$ distinguishes unit-weight graph families. (c) The Kirchhoff index separates thermal-size classes that the spanning-tree factor alone cannot resolve.

G controls internal free energy and generalized bond forces. Dynamically, free center-of-mass kinetic modes and interacting internal modes generally relax at different rates, so their total energy follows the closure criterion of Eq. (168). Recent work on interacting levitated nanoparticles studies a hybrid quantum–classical interaction setting [95].

E. Nonlinear graph cage: an exactly solvable collective interaction

The additive model (172) leaves the general factor $\mathcal{Z}_q(G)$ implicit. A complementary family makes the complete graph dependence analytic at every $q > 0$. On the center-of-mass-free subspace define the graph strain and collective power potential

$$S_G(\mathbf{r}) := \mathbf{r}^\top (L_G \otimes I_{d_s}) \mathbf{r} = \sum_{e=(ij)} g_e |\mathbf{r}_i - \mathbf{r}_j|^2, \quad U_{G,q}^{\text{cage}}(\mathbf{r}; \kappa) = \frac{\kappa}{q} S_G(\mathbf{r})^{q/2}. \quad (194)$$

For $q = 2$ this is the harmonic network. The quartic member is a genuine many-body interaction,

$$U_{G,4}^{\text{cage}} = \frac{\kappa}{4} \sum_e g_e^2 |\Delta \mathbf{r}_e|^4 + \frac{\kappa}{2} \sum_{e < f} g_e g_f |\Delta \mathbf{r}_e|^2 |\Delta \mathbf{r}_f|^2, \quad \mathbf{F} = -\kappa S_G(L_G \otimes I_{d_s}) \mathbf{r}. \quad (195)$$

Stretching one edge therefore changes the restoring force on every edge. Equation (194) is a synthetic collective Hamiltonian with a nonadditive force structure. Its equilibrium coordinate law is an elliptically contoured multivariate power-exponential distribution; the

normalization and radial representation of that statistical family are established [96, 97]. We use those identities to build a solvable stochastic-thermodynamic example.

Whitening with the positive restriction of L_G gives, for a connected graph,

$$\begin{aligned}\Omega_{\text{cage}}(E, \kappa, G) &= C_{D,q,M} \kappa^{-D/q} [\det' L_G]^{-d_s/2} E^\alpha, \quad D = d_s(N-1), \\ \alpha &= D \left(\frac{1}{2} + \frac{1}{q} \right), \quad f_{\text{eff}} = 2\alpha, \quad \Gamma_q = \frac{2q}{q+2} \gamma,\end{aligned}\tag{196}$$

where $C_{D,q,M}$ contains only D, q , the internal mass determinant, and the phase-space unit.

The canonical configurational factor is explicitly

$$Z_x(T, \kappa, G) = \frac{2\pi^{D/2}}{q \Gamma(D/2)} \Gamma\left(\frac{D}{q}\right) \left(\frac{q k_B T}{\kappa}\right)^{D/q} [\det' L_G]^{-d_s/2}.\tag{197}$$

The phase-volume control coordinate that reduces Eq. (196) to $\Omega \propto \ell_G^{-\alpha} E^\alpha$ is

$$\frac{\ell_G}{\ell_{G,0}} = \left(\frac{\kappa}{\kappa_0}\right)^{2/(q+2)} \left(\frac{\det' L_G}{\det' L_{G_0}}\right)^{q/[(N-1)(q+2)]}.\tag{198}$$

Thus global stiffness protocols at fixed G inherit every driven-CIR, heat, work, and cycle result derived above.

The determinant factor makes the separation between nonlinear radial physics and graph geometry exact:

$$\begin{aligned}F_G &= F_0(T, \kappa, q, M) + \frac{d_s k_B T}{2} \ln \det' L_G, \\ \mathcal{X}_e^{\text{cage}} &:= \frac{\partial F_G}{\partial g_e} = \frac{d_s k_B T}{2} R_e, \quad \frac{\partial^2 F_G}{\partial g_e \partial g_f} = -\frac{d_s k_B T}{2} (\mathbf{b}_e^\top L_G^+ \mathbf{b}_f)^2.\end{aligned}\tag{199}$$

The response follows from the established matrix-tree and transfer-impedance identities [91, 92, 98, 99]. For $q \neq 2$, the second derivative combines an explicit curvature of $U_{G,q}^{\text{cage}}$ with a force covariance. The sum is the resistance expression in Eq. (199).

The spatial observables are equally explicit. With

$$\mathcal{R}_q^2(T, \kappa) := \left(\frac{q k_B T}{\kappa}\right)^{2/q} \frac{\Gamma[(D+2)/q]}{\Gamma(D/q)},\tag{200}$$

elliptical angular averaging gives

$$\langle |\mathbf{r}_i - \mathbf{r}_j|^2 \rangle = \frac{d_s}{D} \mathcal{R}_q^2 R_{ij}, \quad \langle R_g^2 \rangle = \frac{d_s}{D N^2} \mathcal{R}_q^2 \text{Kf}(G).\tag{201}$$

The exponent q fixes heat capacity, relaxation, energy fluctuations, and the radial scale. The graph fixes relative geometry through the same resistance observables as a harmonic network [100].

F. Exact single-edge graph surgery

The determinant structure also yields an exact nonequilibrium experiment. Prepare the cage canonically with Laplacian L , then change one edge while positions and momenta are frozen,

$$L' = L + \delta g \mathbf{b}_e \mathbf{b}_e^\top, \quad h := \delta g R_e > -1. \quad (202)$$

The inequality preserves confinement. Deleting a bridge gives $h = -1$, changes the nullity, and releases an unconfined coordinate. Let $\beta = (k_B T)^{-1}$. Radial-angular factorization of the initial canonical state gives the exact work representation

$$\begin{aligned} \beta W &= V \left[(1 + hX)^{q/2} - 1 \right], \\ V &\sim \text{Gamma}\left(\frac{D}{q}, 1\right), \quad X \sim \text{Beta}\left(\frac{d_s}{2}, \frac{D - d_s}{2}\right), \quad V \perp X. \end{aligned} \quad (203)$$

Hence one pre-quench number, the effective resistance R_e , carries the full graph dependence of a single-edge work law. Its Laplace transform, all raw moments, mean work, and free-energy change are

$$\begin{aligned} \mathcal{L}_W(s) &:= \langle e^{-s\beta W} \rangle = \mathbb{E}_X \left[1 + s[(1 + hX)^{q/2} - 1] \right]^{-D/q}, \\ \langle (\beta W)^n \rangle &= \left(\frac{D}{q} \right)_n \mathbb{E}_X \left[\{(1 + hX)^{q/2} - 1\}^n \right], \\ \beta \langle W \rangle &= \frac{D}{q} \left[{}_2F_1\left(-\frac{q}{2}, \frac{d_s}{2}; \frac{D}{2}; -h\right) - 1 \right], \quad \beta \Delta F = \frac{d_s}{2} \ln(1 + h). \end{aligned} \quad (204)$$

Here $(a)_n = \Gamma(a + n)/\Gamma(a)$. At $s = 1$, the beta integral collapses to

$$\langle e^{-\beta W} \rangle = \mathbb{E}_X (1 + hX)^{-D/2} = (1 + h)^{-d_s/2} = e^{-\beta \Delta F}, \quad (205)$$

which is the Jarzynski equality and the rank-one determinant update in the same formula [36]. For $q = 2$, beta-gamma algebra reduces βW to a signed gamma variable of shape $d_s/2$ and scale h . Harmonic work quenches are well studied [101, 102]; Eq. (203) is the corresponding thermodynamic parametrization across the collective homogeneous family. The beta law is proper for $N \geq 3$; for the two-vertex graph $D = d_s$ and $X = 1$ deterministically. The confinement condition for the rank-one update is $h > -1$. This is weaker than requiring every individual postquench edge weight to remain nonnegative.

Everything through Eq. (205) holds for arbitrary positive homogeneity degree q . The closed work and entropy-production densities, joint heat-work transform, and engine formulas developed next are the fully equilibrated planar quartic specialization $D = 4$, $q = 4$.

Consider now three particles in two dimensions, start from the unit path P_3 , and add its missing edge with weight g . Then

$$D = 4, \quad q = 4, \quad \alpha = 3, \quad f_{\text{eff}} = 6, \quad \Gamma_4 = \frac{4}{3}\gamma, \quad \frac{\det' L(g)}{\det' L(0)} = 1 + 2g. \quad (206)$$

The equilibrium energy, heat capacity, graph free energy, generalized edge force, and end-to-end size are

$$\begin{aligned} \langle E \rangle &= 3k_{\text{B}}T, & C_V &= 3k_{\text{B}}, \\ \frac{F(g) - F(0)}{k_{\text{B}}T} &= \ln(1 + 2g), & \frac{\mathcal{X}_{13}}{k_{\text{B}}T} &= \frac{2}{1 + 2g}, \\ \langle |\mathbf{r}_1 - \mathbf{r}_3|^2 \rangle &= \frac{\sqrt{\pi}}{1 + 2g} \sqrt{\frac{k_{\text{B}}T}{\kappa}}. \end{aligned} \quad (207)$$

For equal internal masses m , its phase volume and partition function are

$$\Omega_G(E) = \frac{4\pi^4 m^2}{3\hbar^4 \kappa \det' L_G} E^3, \quad Z_G(T) = \frac{8\pi^4 m^2 (k_{\text{B}}T)^3}{\hbar^4 \kappa \det' L_G}. \quad (208)$$

For this quench $V \sim \text{Exp}(1)$ and X is uniform on $[0, 1]$. Writing $A = 1 + 2g$ gives

$$\begin{aligned} w &:= \beta W = V\{[1 + 2gX]^2 - 1\}, \\ \mathcal{L}_g(s) &= \frac{\arctan[A\sqrt{s/(1-s)}] - \arctan[\sqrt{s/(1-s)}]}{2g\sqrt{s(1-s)}}, \quad 0 < s < 1. \end{aligned} \quad (209)$$

The continuous work density and first two cumulants are

$$\begin{aligned} p_g(w) &= \frac{1}{4g} \int_0^{A^2-1} \frac{\exp(-w/a)}{a\sqrt{1+a}} da, \quad w > 0, \\ \kappa_1(w) &= 2g + \frac{4}{3}g^2, \quad \kappa_2(w) = \frac{4g^2}{45}(52g^2 + 120g + 75), \\ \frac{\text{Var } W}{\langle W \rangle^2} &= \frac{52g^2 + 120g + 75}{5(2g + 3)^2}. \end{aligned} \quad (210)$$

Equation (204) supplies every higher cumulant. For the unit insertion $g = 1$,

$$\kappa_1(w) = \frac{10}{3}, \quad \kappa_2(w) = \frac{988}{45}, \quad \kappa_3(w) = \frac{299248}{945}, \quad \kappa_4(w) = \frac{33501472}{4725}. \quad (211)$$

The skewness is $3.07810\dots$ and the excess kurtosis is $14.70866\dots$, displaying a broad, strongly non-Gaussian irreversible work law.

After completing the isothermal process by relaxation at fixed final graph, the dimensionless stochastic entropy production is

$$\Sigma = w - \ln A, \quad p_{\Sigma}(\sigma) = p_g(\sigma + \ln A), \quad \mathcal{L}_{\Sigma}(s) = A^s \mathcal{L}_g(s), \quad \langle e^{-\Sigma} \rangle = 1. \quad (212)$$

Its mean equals the irreversible work in thermal units,

$$\langle \Sigma \rangle = \beta \langle W_{\text{irr}} \rangle = 2g + \frac{4}{3}g^2 - \ln(1 + 2g), \quad (213)$$

and the probability of a negative dissipated-work event is

$$\mathcal{P}(\Sigma < 0) = 1 - \int_0^1 \exp \left[-\frac{\ln(1 + 2g)}{(1 + 2gx)^2 - 1} \right] dx. \quad (214)$$

At $g = 1$ this probability is $0.4144485809\dots$. The bare compression work is nonnegative; the negative event is $W < \Delta F$. The reverse triangle-to-path quench begins in a distinct equilibrium ensemble, and its work density obeys the Crooks relation

$$\frac{p_F(w)}{p_R(-w)} = e^{w - \ln A}. \quad (215)$$

Thus the forward law alone has no same-distribution strong detailed fluctuation theorem [37].

G. Relaxation, protocols, and a graph-surgery engine

The unit-edge quench from P_3 to K_3 raises the mean energy from $3k_B T$ to $19k_B T/3$. Subsequent fixed-graph relaxation is governed, within the leading weak-friction averaged theory, by

$$dE = \Gamma_4(3k_B T - E)dt + \sqrt{2\Gamma_4 k_B T E} dB, \quad \langle Q_t \rangle = -\frac{10}{3}k_B T(1 - e^{-\Gamma_4 t}). \quad (216)$$

Complete relaxation gives

$$\Delta S_{\text{sys}} = -k_B \ln 3, \quad \Delta S_{\text{env}} = \frac{10}{3}k_B, \quad \Delta S_{\text{tot}} = k_B \left(\frac{10}{3} - \ln 3 \right). \quad (217)$$

At finite time, the unrelaxed nonequilibrium free energy supplies the missing part of the completed-process entropy in Eq. (217).

The postquench energy law and its relaxation remain analytic. For $e_+ = \beta E_{0+}$ and $\chi = e^{-\Gamma_4 t}$,

$$L_+(s) := \langle e^{-se_+} \rangle = (1 + s)^{-2} \frac{\arctan(3\sqrt{s}) - \arctan(\sqrt{s})}{2\sqrt{s}},$$

$$\langle e^{-s\beta E_t} \rangle = [1 + s(1 - \chi)]^{-3} L_+ \left(\frac{s\chi}{1 + s(1 - \chi)} \right). \quad (218)$$

Even the joint quench-work and relaxation-heat statistics close. Define $q_t = \beta(E_t - E_{0+})$, $B_\eta = 1 - \eta(1 - \chi)$, and $c_\eta = \eta(1 - \chi)(1 - \eta)/B_\eta$. Then

$$\langle e^{\xi w + \eta q_t} \rangle = B_\eta^{-3} (1 + c_\eta)^{-2} \frac{1}{2} \int_1^3 \frac{dr}{1 + c_\eta r^2 - \xi(r^2 - 1)}, \quad (219)$$

throughout the common convergence domain. This gives every mixed work-heat cumulant by differentiation.

Three protocol families separate cleanly. At fixed graph, arbitrary global $\kappa(t)$ uses $\ell = \kappa^{1/3}$ and retains the scalar driven CIR description. Proportional graph breathing $L(t) = c(t)L_0$ is equivalent to the single stiffness $\kappa_{\text{eff}} = \kappa c^2$ and has

$$\ell_{\text{eff}} = \kappa^{1/3} c^{2/3}, \quad a(t) = \frac{1}{3} \frac{\dot{\kappa}}{\kappa} + \frac{2}{3} \frac{\dot{c}}{c}. \quad (220)$$

For continuous single-edge driving in the triangle,

$$\ell_{\text{DOS}} \propto [\kappa(1 + 2g)]^{1/3}, \quad a_{\text{DOS}} = \frac{1}{3} \frac{\dot{\kappa}}{\kappa} + \frac{2\dot{g}}{3(1 + 2g)}. \quad (221)$$

This last expression is the microcanonical shell-average drift. A finite-rate edge protocol resolves an angular bond observable, so scalar CIR closure requires shell mixing or an augmented state. The sudden law (203) and the quasistatic isothermal work $\Delta F = k_B T \ln(1 + 2g)$ remain exact endpoint results.

A four-stroke graph-surgery cycle follows by inserting the edge in path equilibrium at T_C , thermalizing the triangle at T_H , deleting the edge in triangle equilibrium, and thermalizing the path at T_C . Put $r_T = T_H/T_C$ and define the dimensionless quench means and variances

$$\begin{aligned} \mu_+(g) &= \frac{2g(2g + 3)}{3}, & v_+(g) &= \frac{4g^2(52g^2 + 120g + 75)}{45}, \\ \mu_-(g) &= -\frac{2g(4g + 3)}{3(1 + 2g)^2}, & v_-(g) &= \frac{4g^2(112g^2 + 180g + 75)}{45(1 + 2g)^4}. \end{aligned} \quad (222)$$

Complete equilibration makes the two quenches independent. The cycle thermodynamics is therefore

$$\begin{aligned} \frac{\langle W_{\text{cyc}} \rangle}{k_B T_C} &= \mu_+ + r_T \mu_-, & \frac{\text{Var } W_{\text{cyc}}}{(k_B T_C)^2} &= v_+ + r_T^2 v_-, \\ \frac{\langle Q_H \rangle}{k_B T_C} &= 3r_T - 3 - \mu_+, & \frac{\langle Q_C \rangle}{k_B T_C} &= 3 - r_T(3 + \mu_-), \\ \langle \Sigma_{\text{cyc}} \rangle &= 3 \left(r_T + \frac{1}{r_T} - 2 \right) + \frac{\mu_+}{r_T} + r_T \mu_-. \end{aligned} \quad (223)$$

The device produces work when

$$r_T > r_T^{\text{eng}}(g) := \frac{(1+2g)^2(2g+3)}{4g+3}, \quad \eta_{\text{gs}} = \frac{-\mu_+ - r_T\mu_-}{3r_T - 3 - \mu_+}. \quad (224)$$

The nonnegative entropy in Eq. (223) places η_{gs} below the Carnot value. The complete cycle-work transform is a product of the forward and reverse one-edge transforms,

$$\mathcal{L}_{\text{cyc}}(s) = \mathcal{L}_g(s) \mathcal{L}_g^{\text{rev}}(r_T s), \quad \mathcal{L}_g^{\text{rev}}(s) = \frac{1+2g}{2g} \int_{1/(1+2g)}^1 \frac{dr}{1+s(r^2-1)}. \quad (225)$$

Equations (223)–(225) are a full-equilibration benchmark. Finite thermal contacts correlate successive surgeries and require propagation of the non-gamma law (218) together with its angular sector.

The force in Eq. (195) is a concrete target for measurement-based virtual-potential control. Feedback traps already impose time-dependent synthetic potentials, and multiparticle levitated platforms now support tunable long-range couplings [103–105]. An implementation would measure all particle positions, subtract the center of mass, compute S_G , and apply the common gain κS_G to the graph force. Controller delay and bandwidth then become additional experimental scales. Equation (194) is therefore an experimentally motivated target Hamiltonian; realization of this precise nonlinear feedback network remains open.

XI. HARD-SPHERE SINGULAR LIMIT AND VAN DER WAALS EXTENSION

Hard spheres introduce a fixed length σ_{hs} and therefore sit outside the smooth positive-homogeneous family. They nevertheless preserve a stronger part of the energy-sector universality than a fixed-volume equilibrium treatment suggests. For identical particles in a container of volume $\mathcal{V}$, define the graph-restricted configurational integral

$$\mathcal{Q}_G(\mathcal{V}, \sigma_{\text{hs}}) = \int_{\mathcal{V}^N} \prod_{(ij) \in E(G)} \Theta(|\mathbf{r}_i - \mathbf{r}_j| - \sigma_{\text{hs}}) d^{d_s N} r. \quad (226)$$

The physical monodisperse hard-sphere fluid uses $G = K_N$. Its canonical partition function is

$$Z_N^{\text{hs}} = \frac{(2\pi m k_B T)^{d_s N/2}}{N! \mathfrak{h}^{d_s N}} \mathcal{Q}_{K_N}(\mathcal{V}, \sigma_{\text{hs}}). \quad (227)$$

At fixed container volume, summing the Itô kinetic-energy balances of the $d_s N$ independently thermostatted velocity components gives the following scalar equation: elastic particle and stationary-wall collisions conserve total kinetic energy and therefore do not contribute to

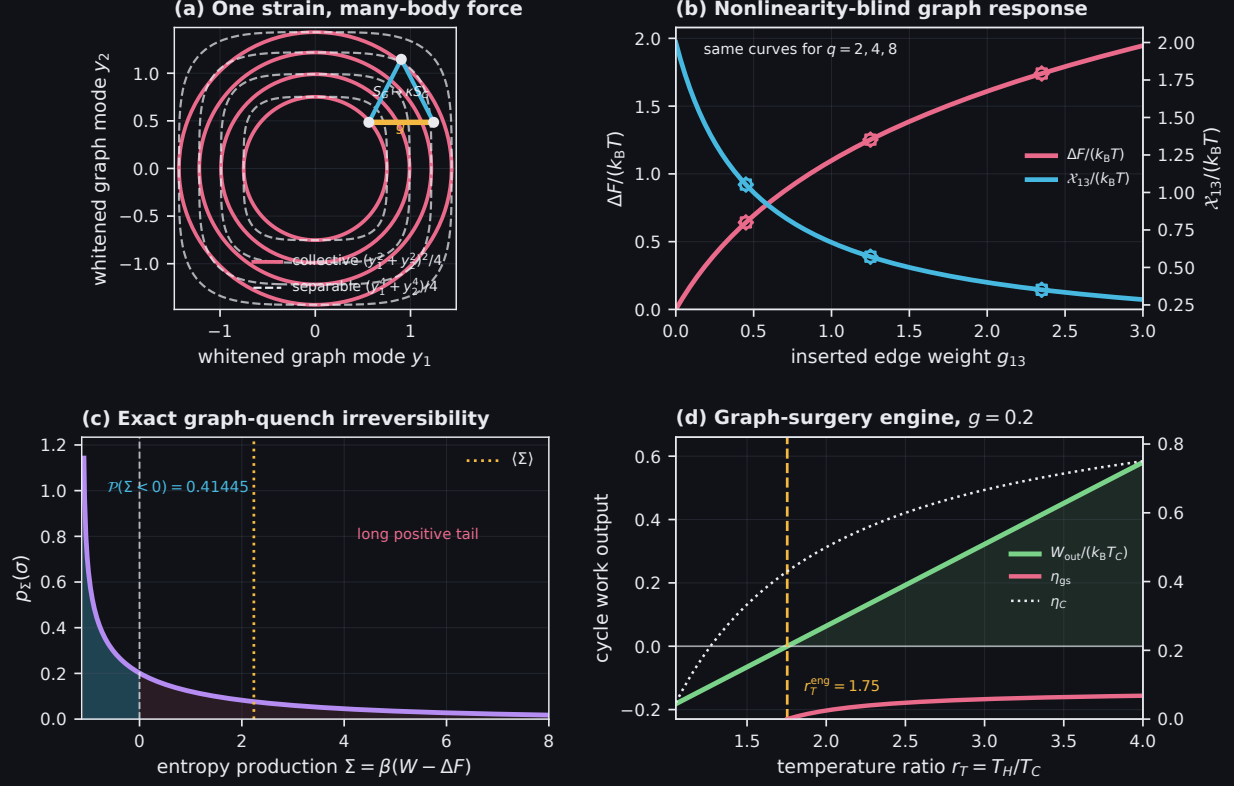


FIG. 10. Collective graph-cage stochastic thermodynamics. (a) The quartic graph cage raises the total graph strain to the second power, producing collective circular contours in whitened graph coordinates; a separable quartic has different level sets. (b) For a tunable planar triangle, the free-energy change and edge force are independent of q and fixed by the spanning-tree polynomial and effective resistance. (c) The unit path-to-triangle quench has an exact entropy-production density with $\mathcal{P}(\Sigma < 0) = 0.41445\dots$; the long positive tail preserves the positive mean and integral fluctuation theorem. (d) A two-temperature graph-surgery cycle produces work above the threshold $r_T^{\text{eng}}(g)$; its efficiency remains below Carnot.

that balance. Event-driven Brownian hard-sphere dynamics is discussed in Ref. [106], but that reference is not used as a derivation of the underdamped energy SDE.

$$dE = 2\gamma \left(\frac{d_s N}{2} k_B T - E \right) dt + \sqrt{4\gamma k_B T E} dB, \quad \alpha_{\text{hs}} = \frac{d_s N}{2}. \quad (228)$$

All fixed-volume heat and exchange-entropy formulas above survive after the substitutions $\alpha \rightarrow \alpha_{\text{hs}}$ and $\Gamma_q \rightarrow 2\gamma$. Excluded volume enters F , S , and p through $\mathcal{Q}_{K_N}$, while

$$\langle E \rangle = \frac{d_s N}{2} k_B T, \quad C_V = \frac{d_s N}{2} k_B. \quad (229)$$

Two distinct steep-potential limits build the confined gas. A positive-power wall confines particles, while the reciprocal potential proposed for the interparticle distance produces an excluded core:

$$\begin{aligned} U_\nu^{\text{wall}}(\mathbf{r}) &= \epsilon_w \sum_{i,a} \left| \frac{2x_{ia}}{L_a} \right|^\nu, \\ U_\nu^{\text{core}}(\mathbf{r}) &= \epsilon_p \sum_{(ij) \in E(G)} \left(\frac{\sigma_{\text{hs}}}{|\mathbf{r}_i - \mathbf{r}_j|} \right)^\nu. \end{aligned} \quad (230)$$

As $\nu \rightarrow \infty$, the first is zero inside the box and infinite outside, whereas

$$\lim_{\nu \rightarrow \infty} \exp \left[-\frac{\epsilon_p}{k_B T} \left(\frac{\sigma_{\text{hs}}}{r} \right)^\nu \right] = \Theta(r - \sigma_{\text{hs}}) \quad (231)$$

away from contact. For one particle in one dimension, the wall limit of $U(x) \propto |x|^{2n}$ has $f_{\text{eff}} \rightarrow 1$ and $\alpha \rightarrow 1/2$, precisely the particle-in-a-box limit of Refs. [17, 18]. The inverse-power expression $(L/|x - x_0|)^{2n}$ instead excludes the interval $|x - x_0| < L$ and is the pairwise route to a hard core. At finite ν , their sum contains positive and negative homogeneities and is not in the single- q gamma class. Convergence of Boltzmann weights also does not alone prove trajectory convergence to reflected hard-sphere dynamics [107].

The moving-boundary construction below is a conditional leading-order closure, not a trajectory-level convergence theorem for singular hard spheres. We assume that fixed-volume shells mix faster than both thermal relaxation and boundary driving, that impulsive wall work has the shell average $-\dot{\lambda}\Omega_\lambda/\omega$, and that no boundary-induced drift, diffusion, or collision-phase variable survives at the retained order. Convergence of equilibrium Boltzmann weights or phase volumes alone does not establish these dynamical assumptions.

Subject to these assumptions, the nonequilibrium connection reappears at leading shell-averaged order in the singular limit. The enclosed phase volume factorizes as

$$\Omega(E, \mathcal{V}, G) = C \mathcal{Q}_G(\mathcal{V}, \sigma_{\text{hs}}) E^\alpha, \quad \alpha = \frac{d_s N}{2}. \quad (232)$$

Subject to these assumptions, for a moving container whose wall work remains in the orbit-averaged regime,

$$\begin{aligned} dE &= [a_{\text{hs}}(t)E - 2\gamma(E - \alpha k_B T)] dt + \sqrt{4\gamma k_B T E} dB, \\ a_{\text{hs}}(t) &= -\frac{1}{\alpha} \frac{d}{dt} \ln \mathcal{Q}_G(\mathcal{V}_t, \sigma_{\text{hs}}), \quad \ell_{\text{hs}} := \mathcal{Q}_G^{-1/\alpha}. \end{aligned} \quad (233)$$

Within this conditional shell-averaged model, Eq. (233) is the same driven CIR process as Eq. (32), with $a_{\text{hs}} = \dot{\ell}_{\text{hs}}/\ell_{\text{hs}}$. The heat laws, Riccati transform, and constrained-control formulas therefore carry over in ℓ_{hs} . At finite wall speed outside this averaging hierarchy, the process need not remain CIR and additional collision-phase or boundary variables may be required. The shell pressure and its thermal mean are

$$p(E, \mathcal{V}) = \frac{E}{\alpha} \partial_{\mathcal{V}} \ln \mathcal{Q}_G, \quad \langle p \rangle = k_{\text{B}} \theta \partial_{\mathcal{V}} \ln \mathcal{Q}_G. \quad (234)$$

This provides a direct nonequilibrium role for the full graph-dependent $\ln \mathcal{Q}_G$, beyond its equilibrium free-energy contribution. Work statistics for moving boxes and interacting classical gases provide related finite-time benchmarks [19, 108].

In three dimensions, with number density $\rho = N/\mathcal{V}$ and packing fraction $\varphi = \pi \rho \sigma_{\text{hs}}^3/6$, the virial expansion begins [109, 110]

$$\begin{aligned} \frac{p}{\rho k_{\text{B}} T} &= 1 + B_2 \rho + B_3 \rho^2 + \dots = 1 + 4\varphi + 10\varphi^2 + \dots, \\ B_2 &= \frac{2\pi}{3} \sigma_{\text{hs}}^3, \quad B_3 = \frac{5\pi^2}{18} \sigma_{\text{hs}}^6. \end{aligned} \quad (235)$$

The Carnahan–Starling resummation and its excess free energy are [111]

$$\frac{p_{\text{CS}}}{\rho k_{\text{B}} T} = \frac{1 + \varphi + \varphi^2 - \varphi^3}{(1 - \varphi)^3}, \quad \frac{F_{\text{ex}}^{\text{CS}}}{N k_{\text{B}} T} = \frac{\varphi(4 - 3\varphi)}{(1 - \varphi)^2}. \quad (236)$$

Writing $Z_{\text{hs}} = p\mathcal{V}/(N k_{\text{B}} T)$ makes the dynamical consequence explicit:

$$a_{\text{hs}} = -\frac{2}{d_s} Z_{\text{hs}}(\varphi) \frac{\dot{\mathcal{V}}}{\mathcal{V}}, \quad a_{\text{hs}}^{\text{CS}} = -\frac{2}{d_s} \frac{1 + \varphi + \varphi^2 - \varphi^3}{(1 - \varphi)^3} \frac{\dot{\mathcal{V}}}{\mathcal{V}}. \quad (237)$$

The work-optimal interior remains exponential in ℓ_{hs} , or $d \ln \mathcal{Q}_G/dt = \text{const}$, but physical compression obeys

$$\frac{\dot{\mathcal{V}}}{\mathcal{V}} = -\frac{d_s}{2} \frac{a_s}{Z_{\text{hs}}(\varphi)}. \quad (238)$$

Excluded-volume correlations therefore slow the optimal physical compression by the compressibility factor while leaving the phase-volume protocol universal.

The same coordinate produces an interacting hard-sphere Otto cycle. Between two volumes $\mathcal{V}_C > \mathcal{V}_H$, its orbit-adiabatic energy ratio is

$$\rho_G = \frac{\ell_{\text{hs}}(\mathcal{V}_H, G)}{\ell_{\text{hs}}(\mathcal{V}_C, G)} = \left[\frac{\mathcal{Q}_G(\mathcal{V}_C, \sigma_{\text{hs}})}{\mathcal{Q}_G(\mathcal{V}_H, \sigma_{\text{hs}})} \right]^{1/\alpha}. \quad (239)$$

Every result in Eqs. (112)–(116) survives with $\rho \rightarrow \rho_G$. In the ideal gas, $\rho_G = (\mathcal{V}_C/\mathcal{V}_H)^{2/d_s}$. Carnahan–Starling gives the explicit interacting correction

$$\rho_{\text{CS}} = \left(\frac{\mathcal{V}_C}{\mathcal{V}_H}\right)^{2/d_s} \exp\left\{\frac{2}{d_s}[f_{\text{ex}}(\varphi_H) - f_{\text{ex}}(\varphi_C)]\right\}, \quad f_{\text{ex}}(\varphi) = \frac{\varphi(4-3\varphi)}{(1-\varphi)^2}. \quad (240)$$

Thus the full graph configurational integral shifts the cycle compression and all work statistics. This is the nonequilibrium observable that depends explicitly on G .

A van der Waals attraction requires a second ingredient. For a hard core plus an integrable weak tail $u_{\text{att}}(r) < 0$, with $|u_{\text{att}}|/(k_B T) \ll 1$ and $\int_{\sigma_{\text{hs}}}^{\infty} |u_{\text{att}}(r)| r^2 dr < \infty$, the exact second-virial formula and its weak-tail expansion are [110]

$$B_2(T) = -2\pi \int_0^{\infty} [e^{-u(r)/(k_B T)} - 1] r^2 dr, \\ B_2(T) \simeq b - \frac{a_{\text{vdW}}}{k_B T}, \quad b = \frac{2\pi}{3} \sigma_{\text{hs}}^3, \quad a_{\text{vdW}} = -2\pi \int_{\sigma_{\text{hs}}}^{\infty} u_{\text{att}}(r) r^2 dr > 0. \quad (241)$$

A standard phenomenological free-volume plus mean-field closure that reproduces this second virial coefficient is

$$p = \frac{\rho k_B T}{1 - b\rho} - a_{\text{vdW}} \rho^2. \quad (242)$$

A closure-preserving nonequilibrium version introduces only the uniform density mean field $U_{\text{mf}} = -a_{\text{vdW}} N^2/\mathcal{V}$. The kinetic energy retains Eq. (233), total work gains dU_{mf} , and

$$\langle p \rangle = k_B \theta \partial_{\mathcal{V}} \ln \mathcal{Q}_G - \frac{a_{\text{vdW}} N^2}{\mathcal{V}^2} \simeq \frac{N k_B \theta}{\mathcal{V} - Nb} - \frac{a_{\text{vdW}} N^2}{\mathcal{V}^2}. \quad (243)$$

The last approximation uses $\mathcal{Q} \simeq (\mathcal{V} - Nb)^N$ and becomes Eq. (242) at equilibrium $\theta = T$. A true Kac-type mean-field limit supplies a controlled route to this form [112]. By contrast, a microscopic attractive tail exchanges kinetic and potential energy while the thermostat noise samples only the kinetic part. It generally destroys scalar CIR closure. Pure hard spheres have $a_{\text{vdW}} = 0$; freezing and the attractive liquid–gas transition require configurational information beyond the scalar energy SDE. For graph-restricted exclusions, $\mathcal{Q}_G$ expands in Mayer subgraphs rather than a Laplacian determinant; graph sparsity and treewidth may organize computation, while the harmonic complete-monotonicity theorem does not transfer automatically.

XII. WHERE UNIVERSALITY ENDS

Equation (13) applies to a general regular Hamiltonian whenever averaging is energy-lumpable. Its coefficients are then nonlinear through Ω/ω , and $f_{\text{eff}}(E, \lambda)$ runs. Gamma

distributions, a single effective temperature, CIR kernels, and scalar Riccati transforms require phase-volume homogeneity. Mixed powers, barriers, separatrices, rest lengths, and local coupling protocols generally break this closure.

The scope assumptions are regular connected energy components over the range used, $\varepsilon \ll 1$, a Markovian momentum bath, and a specified no-flux boundary when $\alpha < 1$. In more than one degree of freedom, shell mixing is one sufficient mechanism; orbitwise identities can also prove energy lumpability. Separatrices, coefficients retaining extra actions, colored noise, nonuniform thermostats, and mixed homogeneity require extensions.

Engine formulas require additional bath and control structure. A cycle coupled to one bath held at fixed temperature is dissipative. The finite-time atlas in Secs. VIII and IX fixes five named stroke architectures and one globally constrained smooth-control class before optimization. Carnot, Otto, and Brayton reach the Curzon–Ahlborn number through distinct optimized variables; Stirling selects an interior contact-time allocation; and unconstrained Ericsson pressure-ratio power approaches a plateau. The smooth quadrature optimum assumes gamma preparation, local fluctuation–dissipation balance, constant Γ_q , and driving slow enough for orbit averaging. Bounds on x , $\dot{x}$, T , or $\dot{T}$ are absent from its quadratic-budget problem; imposing any of them generally changes the optimizer. The physical-gas isobars constrain ensemble-mean pressure, so their trajectory work still requires the Riccati path functional. Shortcuts to isothermality add auxiliary control forces and form a distinct problem [113].

The dimensionless groups that segment experiments are

$$\begin{aligned} \alpha = \frac{f_{\text{eff}}}{2}, \quad \Gamma_q t, \quad \frac{a}{\Gamma_q}, \quad \ln \frac{\lambda_f}{\lambda_i}, \quad \frac{T_C}{T_H}, \quad \frac{\Gamma_C}{\Gamma_H}, \\ \frac{N}{\alpha}, \quad \ln \Pi, \quad \frac{\omega}{\Gamma_q}, \quad \frac{R_\theta}{\omega \bar{\theta}}, \quad M, \quad h = \delta g R_e, \\ \varphi, \quad \frac{a_{\text{vdW}} \rho}{k_B T}. \end{aligned} \tag{244}$$

The first two identify the homogeneous energy class; the next four specify generic driving and thermal operation; N/α and $\ln \Pi$ select physical-gas isobars and their pressure range. The ratio ω/Γ_q , normalized slew $R_\theta/(\omega \bar{\theta})$, and mismatch M specify the smooth cycle. The edge-surgery number h combines graph topology and quench amplitude. The last two quantities measure excluded-volume and attractive-gas nonideality. Interaction networks add the shape coordinates G through $\ln \mathcal{Z}_q(G)$. In the equal-mass harmonic class, $\ln \det' L$, R_{ij} ,

$\text{Kf}(G)$, and λ_2 are respectively free-energy, local-response, size, and soft-mode observables; unequal masses replace λ_2 by the positive generalized stiffness spectrum of (L, M) .

XIII. CONCLUSION

Kramers energy diffusion is the starting point, while stochastic thermodynamics supplies a different set of questions. The phase-volume formulation supplies DOS-relative thermodynamic bookkeeping, separates work transport from irreversible bath current, and reduces positive homogeneous Hamiltonians to a two-parameter energy universality class. Fixed-control gamma relaxation yields a closed total-path entropy-production density. For the established strong-DFT exchange family, the representation $[1 - Ls(1 + s)]^{-\alpha}$ generates the compact cumulant recurrence and exact violation and uncertainty identities derived here.

Within that class, f_{eff} is the thermodynamic size: it fixes the gamma shape, $C_V = f_{\text{eff}}k_B/2$, relative fluctuations, and cumulant scale. The rate Γ_q is the kinetic clock: it fixes relaxation, finite-time dissipation, optimal stroke allocation, and power. The exact exchange identity $V = 2 \langle \Sigma_x \rangle + \langle \Sigma_x \rangle^2 / \alpha$ generates the higher cumulants recursively and becomes an affinity-free heat-current relation. Its additive $1/\alpha$ separation from the parabolic benchmark vanishes for large systems, while the multiplicative gap remains finite at fixed intensive contrast. Minimizing mean production lowers absolute fluctuations but worsens relative precision; it is therefore not an independent Carnot variational principle. The control Hamiltonian places boundary layers, singular isotherms, finite bangs, regularized feedback, and state-boundary arcs in one framework. The six-member atlas shows exactly how further constraints alter the result: Carnot optimizes isothermal contact durations; Otto optimizes its compression ratio and has a reset work-TUR; Stirling has a closed finite-time contact allocation; Ericsson exposes the physical bound needed for a finite pressure optimum; and Brayton has an exact mean-isobar corner map whose optimal pressure ratio depends on α/N . The sixth member optimizes the full smooth waveform: it saturates a quadratic work bound, yields an exact operational-time CIR covariance, and under a fixed temperature span balances logarithmic state slew against bath–state mismatch. This finite-time content complements the known quasistatic phase-volume atlas [61].

The interaction-network extension separates total-energy universality from graph thermodynamic geometry. Harmonic networks expose the latter through spanning-tree free energy,

effective-resistance response, Kirchhoff-index size, and spectral time-scale separation. The nonlinear graph cage retains those observables for every q while coupling all edges through one collective strain. Its single-edge quench has an exact beta–gamma work law controlled by $h = \delta g R_e$; the planar quartic triangle then yields closed work and entropy-production densities, a joint work–heat transform, and a full-equilibration graph-surgery engine. These nonequilibrium results depend on more than the determinant normalization inherited from elliptical integration. Treewidth gives a sufficient complete-monotonicity certificate for analytically continued exponents and organizes sparse elimination, while polynomial and spectral observables resolve the physical graph classes. Continuous shape protocols expose the next frontier because scalar closure can require mixing or extra orbit labels. Subject to rapid shell mixing and orbit-averaged moving-wall work, hard spheres provide a conditional singular extension: the factorization $\Omega = C Q_G E^\alpha$ makes $Q_G^{-1/\alpha}$ the mechanical control, so packing and graph structure enter nonequilibrium pressure, optimal compression, and Otto work statistics while the energy remains CIR at leading shell-averaged order. A microscopic attractive tail generally destroys this scalar closure. A uniform density mean field yields a controlled nonequilibrium van der Waals correction. These separations identify which observables remain universal and exactly where additional state information enters.

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Appendix A: Thermal-manifold checks

Insertion of Eq. (36) into the Fokker–Planck equation gives Eq. (37). On this family,

$$S(\theta, \lambda) = S_0 + \alpha k_B \ln \frac{\theta}{\sqrt{\lambda}}. \quad (\text{A1})$$

Consequently

$$\begin{aligned} \frac{\dot{S}}{k_B} &= \alpha \Gamma_q \left(\frac{T}{\theta} - 1 \right), \\ \frac{\dot{Q}}{k_B T} &= \alpha \Gamma_q \left(1 - \frac{\theta}{T} \right), \end{aligned} \quad (\text{A2})$$

whose difference gives Eq. (39).

Appendix B: Heat transform

The conditional CIR transform is

$$\mathbb{E}[e^{-sE_t} | E_0] = [1 + k_B T s(1 - r_t)]^{-\alpha} \exp \left[-\frac{s r_t E_0}{1 + k_B T s(1 - r_t)} \right]. \quad (\text{B1})$$

Averaging over the initial gamma law gives Eq. (44). In the factorization $Q = \varepsilon_+ G_1 - \varepsilon_- G_2$, the ratio $G_1/(G_1 + G_2)$ is Beta(α, α), which gives Eq. (51).

Appendix C: Thermodynamically consistent energy bins

A discrete equation for probabilities p_n may represent coarse classical energy cells. For bins I_n define

$$\pi_n = Z^{-1} \int_{I_n} \omega(E, \lambda) e^{-E/k_B T} dE. \quad (\text{C1})$$

One positive nearest-neighbor discretization uses

$$k_{n,n+1} = \frac{D_{n+1/2}}{(\Delta E)^2} \sqrt{\frac{\pi_{n+1}}{\pi_n}}, \quad k_{n+1,n} = \frac{D_{n+1/2}}{(\Delta E)^2} \sqrt{\frac{\pi_n}{\pi_{n+1}}}. \quad (\text{C2})$$

It obeys detailed balance, and its dimensionless mean entropy production satisfies

$$\frac{d}{dt} \langle \Sigma_t \rangle_{\text{bin}} = \sum_n J_{n,n+1} \ln \frac{k_{n,n+1} p_n}{k_{n+1,n} p_{n+1}} \geq 0. \quad (\text{C3})$$

These bin jumps are discretization events and have no microscopic transition interpretation.

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- [1] H. A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica* **7**, 284 (1940).
 - [2] P. Hänggi, P. Talkner, and M. Borkovec, Reaction-rate theory: fifty years after Kramers, *Rev. Mod. Phys.* **62**, 251 (1990).
 - [3] B. Carmeli and A. Nitzan, Theory of activated rate processes: Coupled energy diffusion and spatial diffusion, *Phys. Rev. Lett.* **51**, 233 (1983).
 - [4] M. Büttiker, E. P. Harris, and R. Landauer, Thermal activation in extremely underdamped josephson-junction circuits, *Phys. Rev. B* **28**, 1268 (1983).
 - [5] J. E. Straub and B. J. Berne, Energy diffusion in many-dimensional markovian systems: The consequences of competition between inter- and intramolecular vibrational energy transfer, *J. Chem. Phys.* **85**, 2999 (1986).
 - [6] V. I. Mel'nikov and S. V. Meshkov, Theory of activated rate processes: Exact solution of the Kramers problem, *J. Chem. Phys.* **85**, 1018 (1986).
 - [7] E. Pollak, H. Grabert, and P. Hänggi, Theory of activated rate processes for arbitrary frequency dependent friction: Solution of the turnover problem, *J. Chem. Phys.* **91**, 4073 (1989).
 - [8] E. Pollak, P. Talkner, and A. M. Berezhkovskii, A theory for nonisothermal unimolecular reaction rates, *J. Chem. Phys.* **107**, 3542 (1997).
 - [9] S. Linkwitz and H. Grabert, Energy diffusion of a weakly damped and periodically driven particle in an anharmonic potential well, *Phys. Rev. B* **44**, 11888 (1991).
 - [10] C. Jarzynski, Diffusion equation for energy in ergodic adiabatic ensembles, *Phys. Rev. A* **46**, 7498 (1992).
 - [11] C. Jarzynski, Energy diffusion in a chaotic adiabatic billiard gas, *Phys. Rev. E* **48**, 4340 (1993).

- [12] J. Gieseler, L. Novotny, and R. Quidant, Thermal nonlinearities in a nanomechanical oscillator, *Nat. Phys.* **9**, 806 (2013).
- [13] J. Gieseler, R. Quidant, C. Dellago, and L. Novotny, Dynamic relaxation of a levitated nanoparticle from a non-equilibrium steady state, *Nat. Nanotechnol.* **9**, 358 (2014).
- [14] L. Rondin, J. Gieseler, F. Ricci, R. Quidant, C. Dellago, and L. Novotny, Direct measurement of Kramers turnover with a levitated nanoparticle, *Nat. Nanotechnol.* **12**, 1130 (2017).
- [15] J. Gieseler and J. Millen, Levitated nanoparticles for microscopic thermodynamics—a review, *Entropy* **20**, 326 (2018).
- [16] D. S. P. Salazar and S. A. Lira, Exactly solvable nonequilibrium langevin relaxation of a trapped nanoparticle, *J. Phys. A: Math. Theor.* **49**, 465001 (2016), arXiv:1605.02767.
- [17] D. S. P. Salazar and S. A. Lira, Stochastic thermodynamics of nonharmonic oscillators in high vacuum, *Phys. Rev. E* **99**, 062119 (2019), arXiv:1811.02601.
- [18] Z. Gong, Y. Lan, and H. T. Quan, Stochastic thermodynamics of a particle in a box, *Phys. Rev. Lett.* **117**, 180603 (2016).
- [19] D. S. P. Salazar, Work distribution in thermal processes, *Phys. Rev. E* **101**, 030101(R) (2020), arXiv:1911.02099.
- [20] A. M. Timpanaro, G. Guarnieri, J. Goold, and G. T. Landi, Thermodynamic uncertainty relations from exchange fluctuation theorems, *Phys. Rev. Lett.* **123**, 090604 (2019), arXiv:1904.07574.
- [21] D. S. P. Salazar, Detailed fluctuation theorem bound for apparent violations of the second law, *Phys. Rev. E* **104**, L062101 (2021), arXiv:2110.00630.
- [22] D. S. P. Salazar, Thermodynamic skewness relation from detailed fluctuation theorem, *Phys. Rev. E* **106**, L042101 (2022), arXiv:2208.11206.
- [23] D. S. P. Salazar, Bound for the moment generating function from the detailed fluctuation theorem, *Phys. Rev. E* **107**, L062103 (2023), arXiv:2302.02998.
- [24] G. T. Landi and M. Paternostro, Irreversible entropy production: From classical to quantum, *Rev. Mod. Phys.* **93**, 035008 (2021).
- [25] Y. H. Chen, J.-F. Chen, Z. Fei, and H. T. Quan, Microscopic theory of the Curzon–Ahlborn heat engine based on a Brownian particle, *Phys. Rev. E* **106**, 024105 (2022), arXiv:2108.04128.
- [26] J.-F. Chen and H. T. Quan, Optimal control theory for maximum power of Brownian heat engines, *Phys. Rev. E* **110**, L042105 (2024).

- [27] P. Hertz, über die mechanischen grundlagen der thermodynamik, *Ann. Phys.* **338**, 225 (1910).
- [28] M. Campisi, On the mechanical foundations of thermodynamics: the generalized helmholtz theorem, *Stud. Hist. Philos. Mod. Phys.* **36**, 275 (2005).
- [29] K. Sekimoto, Langevin equation and thermodynamics, *Prog. Theor. Phys. Suppl.* **130**, 17 (1998).
- [30] K. Sekimoto, *Stochastic Energetics* (Springer, Berlin, 2010).
- [31] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
- [32] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, 3rd ed. (Springer, Berlin, 2012).
- [33] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods: Averaging and Homogenization* (Springer, New York, 2008).
- [34] M. Esposito, Stochastic thermodynamics under coarse graining, *Phys. Rev. E* **85**, 041125 (2012), [arXiv:1112.5410](#).
- [35] U. Seifert, Entropy production along a stochastic trajectory and an integral fluctuation theorem, *Phys. Rev. Lett.* **95**, 040602 (2005).
- [36] C. Jarzynski, Nonequilibrium equality for free energy differences, *Phys. Rev. Lett.* **78**, 2690 (1997).
- [37] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, *Phys. Rev. E* **60**, 2721 (1999).
- [38] J. C. Cox, J. E. Ingersoll, and S. A. Ross, A theory of the term structure of interest rates, *Econometrica* **53**, 385 (1985).
- [39] C. Jarzynski and D. K. Wójcik, Classical and quantum fluctuation theorems for heat exchange, *Phys. Rev. Lett.* **92**, 230602 (2004).
- [40] R. E. Gaunt, On the moments of the variance-gamma distribution, *Stat. Probab. Lett.* **201**, 109884 (2023).
- [41] P. J. Forrester, On the gamma difference distribution, *Stat. Probab. Lett.* **211**, 110136 (2024).
- [42] A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, *Phys. Rev. Lett.* **114**, 158101 (2015).
- [43] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Dissipation bounds all steady-state current fluctuations, *Phys. Rev. Lett.* **116**, 120601 (2016).

- [44] J. M. Horowitz and T. R. Gingrich, Proof of the finite-time thermodynamic uncertainty relation for steady-state currents, *Phys. Rev. E* **96**, 020103(R) (2017).
- [45] P. Pietzonka, F. Ritort, and U. Seifert, Finite-time generalization of the thermodynamic uncertainty relation, *Phys. Rev. E* **96**, 012101 (2017).
- [46] Y. Hasegawa and T. Van Vu, Fluctuation theorem uncertainty relation, *Phys. Rev. Lett.* **123**, 110602 (2019).
- [47] K. J. Ray, A. B. Boyd, G. Guarnieri, and J. P. Crutchfield, Thermodynamic uncertainty theorem, *Phys. Rev. E* **108**, 054126 (2023).
- [48] T. Speck and U. Seifert, Distribution of work in isothermal nonequilibrium processes, *Phys. Rev. E* **70**, 066112 (2004).
- [49] T. Monnai, Thermodynamic uncertainty relation for quantum work distribution: Exact case study for a perturbed oscillator, *Phys. Rev. E* **105**, 034115 (2022).
- [50] A. Dechant, Multidimensional thermodynamic uncertainty relations, *J. Phys. A: Math. Theor.* **52**, 035001 (2019).
- [51] T. Koyuk and U. Seifert, Thermodynamic uncertainty relation for time-dependent driving, *Phys. Rev. Lett.* **125**, 260604 (2020).
- [52] K. Sekimoto and S.-i. Sasa, Complementarity relation for irreversible process derived from stochastic energetics, *J. Phys. Soc. Jpn.* **66**, 3326 (1997).
- [53] A. Gómez-Marín, T. Schmiedl, and U. Seifert, Optimal protocols for minimal work processes in underdamped stochastic thermodynamics, *J. Chem. Phys.* **129**, 024114 (2008).
- [54] T. Schmiedl and U. Seifert, Optimal finite-time processes in stochastic thermodynamics, *Phys. Rev. Lett.* **98**, 108301 (2007).
- [55] D. A. Sivak and G. E. Crooks, Thermodynamic metrics and optimal paths, *Phys. Rev. Lett.* **108**, 190602 (2012).
- [56] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados, Optimal work in a harmonic trap with bounded stiffness, *Phys. Rev. E* **99**, 012140 (2019), arXiv:1812.09557.
- [57] Z. Ye, F. Cerisola, P. Abiuso, J. Anders, M. Perarnau-Llobet, and V. Holubec, Optimal finite-time heat engines under constrained control, *Phys. Rev. Research* **4**, 043130 (2022).
- [58] A. Zhong and M. R. DeWeese, Limited-control optimal protocols arbitrarily far from equilibrium, *Phys. Rev. E* **106**, 044135 (2022).

- [59] A. Prados, Optimizing the relaxation route with optimal control, *Phys. Rev. Research* **3**, 023128 (2021).
- [60] H. T. Quan, Quantum thermodynamic cycles and quantum heat engines. II, *Phys. Rev. E* **79**, 041129 (2009).
- [61] K. Ito, G.-H. Xu, C. Jiang, É. Roldán, R. A. Rica-Alarcón, I. A. Martínez, and G. Watanabe, Universal relations and bounds for fluctuations in quasistatic small heat engines, *Commun. Phys.* **8**, 60 (2025), arXiv:1910.08096.
- [62] K. Sekimoto, F. Takagi, and T. Hondou, Carnot's cycle for small systems: Irreversibility and cost of operations, *Phys. Rev. E* **62**, 7759 (2000).
- [63] I. A. Martínez, E. Roldán, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Brownian Carnot engine, *Nat. Phys.* **12**, 67 (2016), arXiv:1412.1282.
- [64] Y. Rezek and R. Kosloff, Irreversible performance of a quantum harmonic heat engine, *New J. Phys.* **8**, 83 (2006).
- [65] S. Deffner, Efficiency of harmonic quantum Otto engines at maximal power, *Entropy* **20**, 875 (2018).
- [66] Z. Smith, P. S. Pal, and S. Deffner, Endoreversible Otto engines at maximal power, *J. Non-Equilib. Thermodyn.* **45**, 305 (2020).
- [67] G.-H. Xu, C. Jiang, Y. Minami, and G. Watanabe, Relation between fluctuations and efficiency at maximum power for small heat engines, *Phys. Rev. Research* **4**, 043139 (2022).
- [68] G. S. Agarwal and S. Chaturvedi, Quantum dynamical framework for Brownian heat engines, *Phys. Rev. E* **88**, 012130 (2013).
- [69] V. Blickle and C. Bechinger, Realization of a micrometre-sized stochastic heat engine, *Nat. Phys.* **8**, 143 (2012).
- [70] A. Dechant, N. Kiesel, and E. Lutz, All-optical nanomechanical heat engine, *Phys. Rev. Lett.* **114**, 183602 (2015).
- [71] D. A. Blank and C. Wu, Power limit of an endoreversible Ericsson cycle with regeneration, *Energy Convers. Manage.* **37**, 59 (1996).
- [72] O. Contreras-Vergara, N. Sánchez-Salas, G. Valencia-Ortega, and J. I. Jiménez-Aquino, Carnot, Stirling, and Ericsson stochastic heat engines: Efficiency at maximum power, *Phys. Rev. E* **108**, 014123 (2023).

- [73] C. Wu and R. L. Kiang, Work and power optimization of a finite-time Brayton cycle, *Int. J. Ambient Energy* **11**, 129 (1990).
- [74] M. Bauer, K. Brandner, and U. Seifert, Optimal performance of periodically driven, stochastic heat engines under limited control, *Phys. Rev. E* **93**, 042112 (2016).
- [75] P. A. Erdman, V. Cavina, R. Fazio, F. Taddei, and V. Giovannetti, Maximum power and corresponding efficiency for two-level heat engines and refrigerators: Optimality of fast cycles, *New J. Phys.* **21**, 103049 (2019).
- [76] V. Cavina, P. A. Erdman, P. Abiuso, L. Tolomeo, and V. Giovannetti, Maximum-power heat engines and refrigerators in the fast-driving regime, *Phys. Rev. A* **104**, 032226 (2021).
- [77] G. Jiao, S. Zhu, J. He, Y. Ma, and J. Wang, Fluctuations in irreversible quantum Otto engines, *Phys. Rev. E* **103**, 032130 (2021).
- [78] F. L. Curzon and B. Ahlborn, Efficiency of a Carnot engine at maximum power output, *Am. J. Phys.* **43**, 22 (1975).
- [79] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Efficiency at maximum power of low-dissipation Carnot engines, *Phys. Rev. Lett.* **105**, 150603 (2010), arXiv:1008.2464.
- [80] T. Schmiedl and U. Seifert, Efficiency at maximum power: An analytically solvable model for stochastic heat engines, *Europhys. Lett.* **81**, 20003 (2008).
- [81] C. Li, S. Zhu, P. He, Y. Wang, Y. Zheng, K. Zhang, X. Gao, Y. Dong, and H. Hu, Realization of an all-optical underdamped stochastic Stirling engine, *Phys. Rev. A* **109**, L021502 (2024).
- [82] D. A. Blank, G. W. Davis, and C. Wu, Power optimization of an endoreversible Stirling cycle with regeneration, *Energy* **19**, 125 (1994).
- [83] I. Prieto-Rodríguez, A. Prados, and C. A. Plata, Maximum-power Stirling-like heat engine with a harmonically confined Brownian particle, *Entropy* **27**, 72 (2025).
- [84] J. Kaur, A. Ghosh, S. Dattagupta, S. Chaturvedi, and M. Bandyopadhyay, Stochastic thermodynamics and the Ericsson nano engine—efficiency from equilibrium results, *Physica A* **660**, 130327 (2025).
- [85] Q. Liu, J. He, Y. Ma, and J. Wang, Finite-power performance of quantum heat engines in linear response, *Phys. Rev. E* **100**, 012105 (2019).
- [86] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. (Cambridge University Press, Cambridge, 1952).

- [87] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence* (Wiley, New York, 1986).
- [88] B. E. Eichinger, Configuration statistics of gaussian molecules, *Macromolecules* **13**, 1 (1980).
- [89] T. T. Foley, M. S. Shell, and W. G. Noid, The impact of resolution upon entropy and information in coarse-grained models, *J. Chem. Phys.* **143**, 243104 (2015).
- [90] E. Estrada and N. Hatano, A vibrational approach to node centrality and vulnerability in complex networks, *Physica A* **389**, 3648 (2010).
- [91] N. Biggs, *Algebraic Graph Theory*, 2nd ed. (Cambridge University Press, Cambridge, 1993).
- [92] D. J. Klein and M. Randić, Resistance distance, *J. Math. Chem.* **12**, 81 (1993).
- [93] A. D. Scott and A. D. Sokal, Complete monotonicity for inverse powers of some combinatorially defined polynomials, *Acta Math.* **213**, 323 (2014).
- [94] D. S. P. Salazar, Bounded treewidth and complete monotonicity for scott–sokal spanning-tree polynomials, *arXiv e-prints*, arXiv:2606.26275 (2026), arXiv:2606.26275.
- [95] P. V. Paraguassú and T. Guerreiro, Apparent violations of the second law in the quantum-classical dynamics of interacting levitated nanoparticles, *Phys. Rev. E* **112**, 024134 (2025), arXiv:2501.12305.
- [96] E. Gómez, M. A. Gómez-Villegas, and J. M. Marín, A multivariate generalization of the power exponential family of distributions, *Commun. Stat. Theory Methods* **27**, 589 (1998).
- [97] T. Zhang, A. Wiesel, and M. S. Greco, Multivariate generalized Gaussian distribution: Convexity and graphical models, *IEEE Trans. Signal Process.* **61**, 4141 (2013), arXiv:1304.3206.
- [98] G. Kirchhoff, Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird, *Ann. Phys.* **148**, 497 (1847).
- [99] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, *Ann. Probab.* **21**, 1329 (1993).
- [100] J. Cantarella, T. Deguchi, C. Shonkwiler, and E. Uehara, Radius of gyration, contraction factors, and subdivisions of topological polymers, *J. Phys. A: Math. Theor.* **55**, 475202 (2022).
- [101] C. Kwon, J. D. Noh, and H. Park, Work fluctuations in a time-dependent harmonic potential: Rigorous results and beyond the overdamped limit, *Phys. Rev. E* **88**, 062102 (2013).
- [102] M. Paternostro, G. De Chiara, A. Ferraro, M. Campisi, J. Goold, F. L. Semião, F. Plastina, and V. Vedral, Out of equilibrium thermodynamics of quantum harmonic chains, *J. Stat. Mech.* **2019**, 104014 (2019).

- [103] Y. Jun and J. Bechhoefer, Virtual potentials for feedback traps, *Phys. Rev. E* **86**, 061106 (2012).
- [104] A. Kumar and J. Bechhoefer, Nanoscale virtual potentials using optical tweezers, *Appl. Phys. Lett.* **113**, 183702 (2018), arXiv:1809.02896.
- [105] J. Vijayan, J. Piotrowski, C. Gonzalez-Ballester, K. Weber, O. Romero-Isart, and L. Novotny, Cavity-mediated long-range interactions in levitated optomechanics, *Nat. Phys.* **20**, 859 (2024).
- [106] A. Scala, T. Voigtmann, and C. De Michele, Event-driven Brownian dynamics for hard spheres, *J. Chem. Phys.* **126**, 134109 (2007).
- [107] M. Wilkinson, On the convergence of soft potential dynamics to hard sphere dynamics, *Asymptotic Analysis* **107**, 1 (2018), arXiv:1601.04182.
- [108] G. E. Crooks and C. Jarzynski, Work distribution for the adiabatic compression of a dilute and interacting classical gas, *Phys. Rev. E* **75**, 021116 (2007).
- [109] F. H. Ree and W. G. Hoover, Fifth and sixth virial coefficients for hard spheres and hard disks, *J. Chem. Phys.* **40**, 939 (1964).
- [110] J.-P. Hansen and I. R. McDonald, *Theory of Simple Liquids: With Applications to Soft Matter*, 4th ed. (Academic Press, Oxford, 2013).
- [111] N. F. Carnahan and K. E. Starling, Equation of state for nonattracting rigid spheres, *J. Chem. Phys.* **51**, 635 (1969).
- [112] J. L. Lebowitz and O. Penrose, Rigorous treatment of the Van Der Waals–Maxwell theory of the liquid–vapor transition, *J. Math. Phys.* **7**, 98 (1966).
- [113] G. Li, H. T. Quan, and Z. C. Tu, Shortcuts to isothermality and nonequilibrium work relations, *Phys. Rev. E* **96**, 012144 (2017), arXiv:1608.07757.