Exponentially Accurate Work Extraction by Swapping Detuned Bosonic Modes

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A temperature imbalance between two thermal oscillators can yield exponentially little work, even under arbitrary unitary control. We determine when frequency detuning makes a simple mode swap nearly optimal, by bounding the entire extra work available beyond Gaussian operations. With detuning proportional to the temperature bias near equilibrium, this absolute improvement is exponentially small in the inverse bias. The bound separates exponentially weak extraction at resonance from a detuning window with quadratic work, where the swap captures the leading resource. In this limit, within that window, the accessible fraction of the energy excess above a Gibbs state of equal entropy follows a temperature-independent parabola in detuning normalized by the bias. Inside the window, any fixed fraction below 100% of the leading work can be extracted with an occupation cutoff independent of the bias. The finite bound determines when more general control offers negligible additional work.

Introduction.— A temperature difference provides a source of work, but a small quantum device cannot necessarily release all the energy allowed by thermodynamics. A cyclic unitary preserves every eigenvalue of its initial state, whereas entropy fixes only one function of those eigenvalues. This distinction underlies passivity and ergotropy, the maximum average work obtainable by a unitary [1–3]. It separates work from a single device from collective extraction on many copies [4–6]. Correlations and coherence affect ergotropy [7, 8], while available control shapes thermodynamic performance [9–11]. This isolated-unitary setting differs from bath-assisted work extraction [12, 13] and thermal state conversion [14, 15], including restricted Gaussian thermal operations [16].

Harmonic oscillators make the role of control especially concrete. Gaussian operations, generated by Hamiltonians at most quadratic in the oscillator variables, are central to continuous-variable experiments [17, 18]. Their work-extraction limits have been characterized [19–22]. For two thermal modes, the best Gaussian operation is simply a full mode swap, when that swap lowers the energy, and the identity otherwise. Mode-swap engines and nonlinear extensions are established [23–26]. The remaining question is quantitative: how much work can an arbitrary, possibly non-Gaussian, operation extract beyond the swap?

Equal-frequency oscillators at unequal temperatures already display the difficulty. Their joint state permits work extraction, but accessing it can require transitions between highly excited states [19, 27, 28]. Gaussian control extracts no work there. We show that a frequency mismatch comparable to the temperature imbalance changes this behavior: a mode swap extracts a finite fraction of the thermodynamic energy bound, while the entire advantage of arbitrary control is exponentially small. The new ingredient is a finite bound on that advantage, Eq. (8). It goes beyond the known swap work and the entropy-based estimates of non-Gaussian work potential [22, 29].

We consider two oscillators at temperatures $T_h > T_c$, with excitation energies $a := \hbar\omega_h$ and $c := \hbar\omega_c$ and occupation-number operators $\hat{n}_h, \hat{n}_c$. Write $\beta_j := (k_{\mathrm{B}}T_j)^{-1}$ and $\beta := (\beta_h+\beta_c)/2$. The relative temperature imbalance $\varepsilon := (T_h - T_c)/(T_h + T_c)$ gives $\beta_h = \beta(1-\varepsilon)$ and $\beta_c = \beta(1+\varepsilon)$, with $0 < \varepsilon < 1$.

To measure the frequency mismatch on the same scale, define the *normalized detuning*

$$\delta := \frac{\omega_h/\omega_c - 1}{T_h/T_c - 1}. \tag{1}$$

It locates the frequency ratio between the two boundaries of the swap's operating interval: $\delta = 0$ is equal frequency, $\delta = 1$ is equal mean occupation, and $0 < \delta < 1$ permits positive swap work. For instance, $\delta = 1/2$ is the midpoint of that interval. Let τ_x be the thermal state with occupation probabilities $(1-e^{-x})e^{-xn}$, $n = 0, 1, \ldots$. Our device family is

$$H = a\hat{n}_h + c\hat{n}_c, \quad \rho_\varepsilon = \tau_{\beta_h a} \otimes \tau_{\beta_c c}, \quad \frac{a}{c} = 1 + \frac{2\delta\varepsilon}{1-\varepsilon}. \tag{2}$$

We take $\varepsilon \downarrow 0$ at fixed δ, c, and dimensionless inverse-temperature scale $b := \beta c$. Thus both temperature and frequency differences vanish, while their relative position in the operating interval stays fixed. Each member has a fixed Hamiltonian during extraction; tuning between preparations has a separate work cost. We use $r(x) := (e^x - 1)^{-1}$ for the thermal mean occupation.

Define $\mathcal{W} := \mathrm{tr}(H\rho) - \inf_U \mathrm{tr}(HU\rho U^\dagger)$, with cyclic unitary control U, and let W_{G} be the maximum over Gaussian unitaries. *First main result.*— For the physical family in Eq. (2), the full optimum exceeds the Gaussian optimum by at most an exponentially small amount:

$$0 \le \mathcal{W} - W_{\mathrm{G}} = O_{b,c,\delta}\left(\varepsilon^{-2} e^{-b/[2\max\{\delta,1-\delta\}\varepsilon]}\right). \tag{3}$$

The implicit constant can depend on the fixed parameters b, c, δ. This bounds an absolute work difference. Inside $0 < \delta < 1$, the swap work is quadratic in ε, so its relative

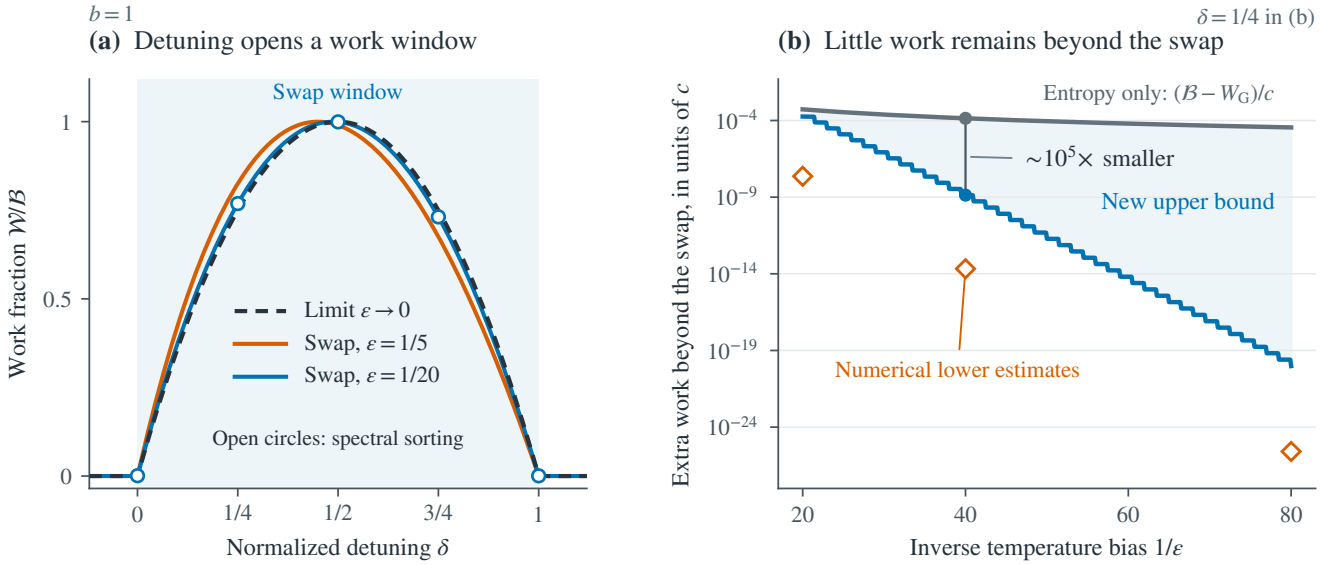


FIG. 1. Detuning makes swap work useful and leaves little room for improvement, at $b = \beta c = 1$, with $c = \hbar\omega_c$. The relative temperature bias is $\varepsilon = (T_h - T_c)/(T_h + T_c)$ and the normalized detuning is $\delta = (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$. (a) The dashed curve is the limiting full-work fraction $\max\{0, 4\delta(1-\delta)\}$. Solid curves show the achievable swap fraction $W_G/\mathcal{B}$, with $\mathcal{B}$ the entropy-matched bound. Open circles are numerical lower estimates of $\mathcal{W}/\mathcal{B}$ from passive sorting at $\varepsilon = 1/20$. (b) At fixed $\delta = 1/4$, the gray entropy-only allowance $(\mathcal{B} - W_G)/c$ is compared with the blue finite upper bound from Eq. (8). Its steps reflect the integer shell cutoff. At $\varepsilon = 1/40$, the marked comparison is 1.3843×10^{-4} versus 1.3513×10^{-9}, about 10^5 times smaller. Open diamonds are numerical lower estimates of $(\mathcal{W} - W_G)/c$, not additional upper bounds. Analytic omitted-tail contributions are below the marker size; numerical precision checks and their limits are given in the SM.

error also vanishes. At and outside the boundaries, $W_G = 0$; Eq. (3) then bounds the entire work obtainable by arbitrary control. We give a finite-parameter version and its proof below.

Second main result.— Detuning controls how much of the thermodynamic energy bound is actually accessible. Let $\mathcal{B} := \mathrm{tr}(H\rho) - \mathrm{tr}(H\gamma_{\beta_*})$, where the Gibbs state $\gamma_{\beta_*} \propto e^{-\beta_* H}$ has the same entropy as ρ. Every unitary preserves entropy, hence $\mathcal{W} \le \mathcal{B}$ [4, 29]. The same-entropy Gibbs bound is established [30, 31]; its attainability on this unitary orbit is the issue here. For Eq. (2), $\mathcal{B} = bc\,r(b)[r(b)+1]\varepsilon^2 + O(\varepsilon^3)$. The known swap work has $W_G/\mathcal{B} \to 4\delta(1-\delta)$ inside the interval. Equation (3) makes the additional work negligible on this quadratic scale, giving

$$\lim_{\varepsilon\downarrow 0} \frac{\mathcal{W}(\rho_\varepsilon)}{\mathcal{B}(\rho_\varepsilon)} = \begin{cases} 4\delta(1-\delta), & 0 < \delta < 1, \\ 0, & \delta \le 0 \text{ or } \delta \ge 1. \end{cases} \tag{4}$$

This is a fraction of the entropy-based energy bound, not of the initial energy. It is independent of b, c within this two-mode joint limit. The midpoint recovers the whole bound to leading order, whereas at resonance the accessible fraction vanishes.

A zero fraction in Eq. (4) does not mean zero work. At and outside the boundaries, we construct two-state swaps whose extracted work matches the exponential scale allowed by Eq. (3). Together the upper and lower bounds give

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln \frac{\mathcal{W}(\rho_\varepsilon)}{c} = -\frac{b}{2\max\{\delta, 1-\delta\}}, \qquad \delta \notin (0,1). \tag{5}$$

Equations (4) and (5) describe the two work regimes of the same device family: quadratic work captured by a swap inside the interval, and exponentially small work even under unrestricted control outside it (Fig. 1a; exterior rates in the SM). We then show how this change reduces the occupation support needed for extraction.

Formalism.— First allow arbitrary excitation energies $a, c > 0$ and dimensionless thermal exponents $x, y > 0$ in $H = a\hat{n}_h + c\hat{n}_c$ and $\rho = \tau_x \otimes \tau_y$. The Gaussian optimum is [19, 22]

$$W_G = \max\{0, (a-c)[r(x) - r(y)]\}. \tag{6}$$

It is achieved by the identity or a full mode swap. Put $e_- := \min(a,c)$, $e_+ := \max(a,c)$, $q := \min(x,y)$, $D := |x-y|$, $C := (1-e^{-x})(1-e^{-y})$, and

$$K := \left\lfloor \min\left\{ \frac{e_-}{e_+ - e_-}, \frac{q}{D} \right\} \right\rfloor. \tag{7}$$

A zero denominator means $+\infty$. If both vanish the state is Gibbs and $\mathcal{W} = W_G = 0$. With occupations m, n and

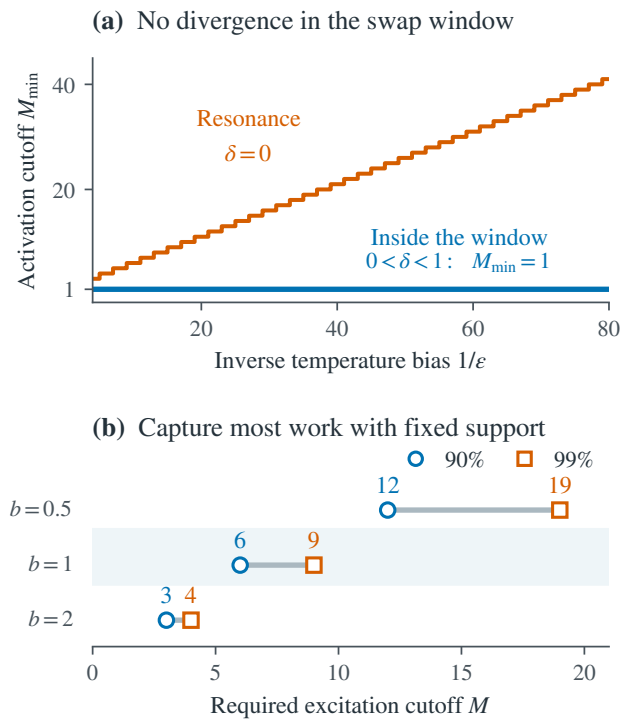


FIG. 2. Detuning changes the occupation support required for extraction. Here $b = \beta\hbar\omega_c$, ε is the relative temperature bias, and δ is the normalized detuning. Controls act on Fock states with total occupation $m + n \le M$ and as identity elsewhere. (a) The least cutoff for positive work diverges at resonance, $M_{\min} = \lfloor (1-\varepsilon)/(2\varepsilon) \rfloor + 2$, while it equals one throughout $0 < \delta < 1$. (b) For fixed interior δ, circles and squares give the least cutoffs for which the limiting fraction $\lim_{\varepsilon \downarrow 0} \mathcal{W}_M/\mathcal{W}$ in Eq. (14) reaches 90% and 99%. The shaded $b = 1$ row gives $M = 6$ and $M = 9$, independent of the vanishing bias. A cutoff-controlled swap need not be Gaussian. The resource shown is occupation support, not gate time or power.

total occupation $N := m + n$, the all-unitary remainder satisfies

$$0 \le \mathcal{W} - W_{\mathrm{G}} \le C e_+ \sum_{N=K+1}^{\infty} N(N+1) e^{-qN}. \qquad (8)$$

The sum has an immediately evaluable geometric form, Eq. (S10) in the Supplemental Material (SM) [32]. Equation (8) is finite-parameter and valid at energy and probability ties.

The proof identifies the part of the spectrum that no further control can improve. Apply the work-extracting swap, if available, and denote the output by σ. Within each shell $N = m + n$, its populations decrease with increasing energy. Every energy in shell N precedes every energy in shell $N+1$ if $(e_+ - e_-)N \le e_-$. Every population in shell N exceeds every population in shell $N+1$ if $DN \le q$. The same two inequalities order shell N relative to every later shell. Hence shells through K occupy their passive spectral positions. The energy in the remaining tail bounds all further work; its shell degeneracy is $N+1$ and each population is at most Ce^{-qN}. Finally, passive energy is invariant under the preliminary swap, so $\mathcal{W}(\rho) = W_{\mathrm{G}} + \mathcal{W}(\sigma)$. The convergent tail controls the infinite-dimensional limit.

For Eq. (2), $x = b[1 + (2\delta - 1)\varepsilon]$ and $y = b(1+\varepsilon)$. Thus $K = [2\max\{\delta, 1-\delta\}\varepsilon]^{-1} + O(1)$, and the geometric tail in Eq. (8) proves Eq. (3). Inside $0 < \delta < 1$, expansion of Eq. (6) and entropy matching give Eq. (4). Outside, swaps between $|u, 0\rangle$ and $|0, v\rangle$, with positive integer occupations $u, v = O(\varepsilon^{-1})$, attain the rate in Eq. (5). Taking one extra integer step avoids an anomalously small prefactor near a tie (SM). The entropy estimate $\mathcal{W} - W_{\mathrm{G}} \le \mathcal{B} - W_{\mathrm{G}}$ permits a much larger residual: at fixed $0 < \delta < 1$,

$$\mathcal{B} - W_{\mathrm{G}} = bc\, r(b)[r(b)+1](1-2\delta)^2\varepsilon^2 + O(\varepsilon^3). \qquad (9)$$

Away from $\delta = 1/2$, this upper bound is quadratic; Eq. (8) replaces it by an exponentially small error.

Applications.— Work extraction from two thermal subsystems is a standard engine setting [30]. Multilevel swap engines [33] and harmonic Otto cycles [34] also include thermalization or changes of level spacing; here we analyze the fixed-Hamiltonian extraction stroke. The interval $0 < \delta < 1$ introduced above is equivalently

$$1 < \frac{\omega_h}{\omega_c} < \frac{\beta_c}{\beta_h} = \frac{T_h}{T_c}. \qquad (10)$$

Inside this window, the lower-frequency mode is colder. In the resonant frequency-conversion model [24], a driven quadratic interaction can implement the swap [35, 36]:

$$H_{\mathrm{ctrl}}(s) = \hbar g(s)\left[e^{-i(\omega_h - \omega_c)s}\hat{a}_h^\dagger \hat{a}_c + \mathrm{H.c.}\right]. \qquad (11)$$

Here s is stroke time, $g(s)$ is a real coupling envelope, $\hat{a}_j$ annihilates one excitation, and H.c. denotes the Hermitian conjugate. In the interaction picture, a pulse with $\int g(s)\, ds = \pi/2$ maps $|m, n\rangle$ to $|n, m\rangle$ up to phases. The coupling vanishes at the endpoints, so the energy decrease of H is the average work transferred to the cyclic control. No claim about its optimal duration is needed.

The operating window and swap work are known; Eq. (8) bounds the extra work available to arbitrary non-Gaussian control. For example, at $b = 1$, $\varepsilon = 1/40$, and $\delta = 1/4$, the frequency ratio is $79/78$ and $W_{\mathrm{G}}/c = 4.369 \times 10^{-4}$. The entropy estimate allows an extra $(\mathcal{B} - W_{\mathrm{G}})/c = 1.3843 \times 10^{-4}$, whereas Eq. (8) bounds it by 1.3513×10^{-9}: less than 3.10×10^{-6} of the swap work. There is also a simple exact example:

$$\frac{\omega_h}{\omega_c} = \sqrt{\frac{\beta_c}{\beta_h}} \quad \Longrightarrow \quad \mathcal{W} = W_{\mathrm{G}} = \mathcal{B}. \qquad (12)$$

After the swap, both physical inverse temperatures equal $\sqrt{\beta_h \beta_c}$, so the output is Gibbs. This is an elementary

saturation condition of the known swap protocol, not an independent optimization over changing Hamiltonians. In the scaled coordinate it is $\delta_*(\varepsilon) = [\sqrt{1-\varepsilon^2} - (1-\varepsilon)]/(2\varepsilon) \to 1/2$.

Occupation support.— High-occupation activation is known [19]; here the exact activation cutoff and captured-fraction law quantify how detuning changes this resource. For a nonnegative integer cutoff M, let $\mathcal{W}_M$ optimize controls on $m+n \le M$, acting as identity on its complement. At resonance,

$$M_{\min} = \left\lfloor \frac{1-\varepsilon}{2\varepsilon} \right\rfloor + 2, \tag{13}$$

where $M_{\min}$ is the least integer permitting positive work. Inside Eq. (10), $M_{\min} = 1$: the single-excitation exchange already yields quadratic work. More strongly, at fixed M and $0 < \delta < 1$,

$$\lim_{\varepsilon\downarrow 0} \frac{\mathcal{W}_M}{\mathcal{W}} = (1-z)^4 \sum_{N=1}^{M} \binom{N+2}{3} z^{N-1}, \qquad z := e^{-b}. \tag{14}$$

The right side increases to one. Any fraction below one can therefore be captured with a cutoff independent of the vanishing bias (Fig. 2). At $b = 1$, $M = 6$ captures over 90% of the leading work, and $M = 9$ over 99%. For sufficiently small ε, reversing occupations within each retained shell is the exact restricted optimum; summing its work gives Eq. (14) (SM). This cutoff-controlled swap need not itself be Gaussian. The unrestricted full swap is Gaussian and needs no tail-selective control.

Experiments have operated single-ion heat engines [37] and characterized energy and ergotropy in oscillator loads [38, 39]. These predictions can be tested in independently prepared harmonic modes, with calibrated frequencies and thermal occupations. Number-state preparation and oscillator control have been demonstrated in circuit systems [40–42], including logical oscillator gates [43] and more general exchange operations [44]; a realization of the present detuning test is proposed, not reported here. On separate ensembles, occupation measurements before and after the swap give records (m_h, m_c) and an average energy decrease $a(\bar{m}_h^{\rm in} - \bar{m}_h^{\rm out}) + c(\bar{m}_c^{\rm in} - \bar{m}_c^{\rm out})$. Here bars denote sample means. Under the thermal-product model, positive sample means give estimates $\hat{x} = \ln(1 + 1/\bar{m}_h^{\rm in})$ and $\hat{y} = \ln(1 + 1/\bar{m}_c^{\rm in})$. A model-based upper bound must maximize the right-hand side of Eq. (8) over a joint confidence region for the mean occupations and frequencies; all-zero samples require occupation bounds instead of these estimates. This compares measured swap work with the largest possible improvement by any unitary. It does not directly measure that optimum or establish Gaussianity from means alone.

The same result bounds the work beyond Gaussian operations for full-rank, finite-energy two-mode Gaussian batteries [17, 21, 22]. A Gaussian unitary brings such a state ρ to a thermal product $\sigma_{\rm G}$ from which Gaussian operations cannot extract further work. Invariance of the full passive energy gives exactly

$$\mathcal{W}(\rho) - W_{\rm G}(\rho) = \mathcal{W}(\sigma_{\rm G}). \tag{15}$$

Apply Eq. (8) to this product, for which $W_{\rm G}(\sigma_{\rm G}) = 0$. The result bounds the non-Gaussian advantage even when the input contains displacements, squeezing, or correlations. Covariance data determine the Gaussian normal form under an assumed Gaussian-state model; no state-independent inference from second moments is implied.

Discussion.— The separation is controlled by ordering of energy shells and populations. At resonance, all low shells are already passive and only a rare tail can yield work. Detuning within Eq. (10) reverses occupation ordering inside populated shells; the mode swap uses that resource, and the residual remains confined to a rare tail. The smooth entropy bound alone misses this transition. Qualitative passivity and virtual-temperature criteria [27, 28] identify possible extraction, while the present bounds quantify both its globally optimized scale and the error of a restricted control family.

The joint limit matters. Fixed nonzero frequency mismatch as $\varepsilon \to 0$ is outside the fixed-δ statement and can involve different spectral arithmetic. Approaching a window edge along an ε-dependent path can also give intermediate powers: $a/c = 1 + 2\varepsilon$ corresponds to $\delta = 1 - \varepsilon$ and cubic swap work, whereas the exact edge $a/c = (1+\varepsilon)/(1-\varepsilon)$ has exponential work. Equation (8) remains valid on both paths. No universal statement about arbitrary fabrication errors follows from exact resonance. Likewise, an occupation cutoff is not a gate-time or power bound, and any cost of frequency tuning between devices or preparations must be accounted for separately.

Numerical checks sort spectra with exact integer energy and probability keys for 84 parameter choices, using two high decimal precisions and analytic tails. The grid includes both window edges and both exterior regions (Fig. S1 in the SM). These checks support the finite bounds without replacing their proofs. Collective extraction from a growing number of copies is a separate limit [4, 6]; no uniform growing-array claim is made here.

Conclusion.— Frequency detuning within the operating window converts an exponentially weak finite-device resource into work accessible by a simple swap. A finite all-unitary remainder bound yields the accessible-work curve, establishes Gaussian near-optimality, and quantifies the change from a diverging activation cutoff to fixed-support extraction. The operating window and its exact Gibbs-producing point turn the resonance obstruction into a usable design criterion.

[1] W. Pusz and S. L. Woronowicz, Passive states and KMS states for general quantum systems, Commun. Math. Phys. **58**, 273 (1978).

[2] A. Lenard, Thermodynamical proof of the Gibbs formula for elementary quantum systems, J. Stat. Phys. **19**, 575 (1978).

[3] A. E. Allahverdyan, R. Balian, and Th. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, Europhys. Lett. **67**, 565 (2004).

[4] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, Phys. Rev. E **87**, 042123 (2013).

[5] K. V. Hovhannisyan, M. Perarnau-Llobet, M. Huber, and A. Acín, Entanglement generation is not necessary for optimal work extraction, Phys. Rev. Lett. **111**, 240401 (2013).

[6] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Extractable work, the role of correlations, and asymptotic freedom in quantum batteries, Phys. Rev. Lett. **122**, 047702 (2019).

[7] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, Extractable work from correlations, Phys. Rev. X **5**, 041011 (2015).

[8] G. Francica, F. C. Binder, G. Guarnieri, M. T. Mitchison, J. Goold, and F. Plastina, Quantum coherence and ergotropy, Phys. Rev. Lett. **125**, 180603 (2020).

[9] H. Wilming, R. Gallego, and J. Eisert, Second law of thermodynamics under control restrictions, Phys. Rev. E **93**, 042126 (2016).

[10] J. Lekscha, H. Wilming, J. Eisert, and R. Gallego, Quantum thermodynamics with local control, Phys. Rev. E **97**, 022142 (2018).

[11] F. Campaioli, S. Gherardini, J. Q. Quach, M. Polini, and G. M. Andolina, Colloquium: Quantum batteries, Rev. Mod. Phys. **96**, 031001 (2024).

[12] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, Nat. Commun. **4**, 2059 (2013).

[13] P. Skrzypczyk, A. J. Short, and S. Popescu, Work extraction and thermodynamics for individual quantum systems, Nat. Commun. **5**, 4185 (2014).

[14] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, Phys. Rev. Lett. **111**, 250404 (2013).

[15] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, Proc. Natl. Acad. Sci. U.S.A. **112**, 3275 (2015).

[16] A. Serafini, M. Lostaglio, S. Longden, U. Shackerley-Bennett, C.-Y. Hsieh, and G. Adesso, Gaussian thermal operations and the limits of algorithmic cooling, Phys. Rev. Lett. **124**, 010602 (2020).

[17] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[18] G. Adesso, S. Ragy, and A. R. Lee, Continuous variable quantum information: Gaussian states and beyond, Open Syst. Inf. Dyn. **21**, 1440001 (2014).

[19] E. G. Brown, N. Friis, and M. Huber, Passivity and practical work extraction using Gaussian operations, New J. Phys. **18**, 113028 (2016).

[20] U. Singh, M. G. Jabbour, Z. Van Herstraeten, and N. J. Cerf, Quantum thermodynamics in a multipartite setting: A resource theory of local Gaussian work extraction for multimode bosonic systems, Phys. Rev. A **100**, 042104 (2019).

[21] N. Friis and M. Huber, Precision and work fluctuations in Gaussian battery charging, Quantum **2**, 61 (2018).

[22] F. E. Quintela Rodriguez, F. A. Mele, S. F. E. Oliviero, V. Giovannetti, L. Lami, and V. Cavina, Extracting energy via bosonic Gaussian operations, arXiv:2503.21748 (2025).

[23] C. Macchiavello, A. Riccardi, and M. F. Sacchi, Quantum thermodynamics of two bosonic systems, Phys. Rev. A **101**, 062326 (2020).

[24] M. F. Sacchi, Thermodynamic uncertainty relations for bosonic Otto engines, Phys. Rev. E **103**, 012111 (2021).

[25] Giovanni Chesi, Chiara Macchiavello, and Massimiliano Federico Sacchi, Work Fluctuations in Ergotropic Heat Engines, Entropy **25**, 1528 (2023).

[26] Giovanni Chesi, Chiara Macchiavello, and Massimiliano Federico Sacchi, Bosonic two-stroke heat engines with polynomial nonlinear coupling, Quantum Sci. Technol. **10**, 035012 (2025).

[27] Paul Skrzypczyk, Ralph Silva, and Nicolas Brunner, Passivity, complete passivity, and virtual temperatures, Phys. Rev. E **91**, 052133 (2015).

[28] Carlo Sparaciari, David Jennings, and Jonathan Oppenheim, Energetic instability of passive states in thermodynamics, Nat. Commun. **8**, 1895 (2017).

[29] A. Sone and S. Deffner, Quantum and classical ergotropy from relative entropies, Entropy **23**, 1107 (2021).

[30] A. E. Allahverdyan, R. S. Johal, and G. Mahler, Work extremum principle: Structure and function of quantum heat engines, Phys. Rev. E **77**, 041118 (2008).

[31] M. N. Bera, A. Riera, M. Lewenstein, Z. B. Khanian, and A. Winter, Thermodynamics as a consequence of information conservation, Quantum **3**, 121 (2019).

[32] See Supplemental Material for complete detuning bounds and proofs, finite-support extraction, Gaussian-orbit extension, and exact-ordering numerical checks with analytic tail control.

[33] R. Uzdin and R. Kosloff, The multilevel four-stroke swap engine and its environment, New J. Phys. **16**, 095003 (2014).

[34] R. Kosloff and Y. Rezek, The quantum harmonic Otto cycle, Entropy **19**, 136 (2017).

[35] Y. Y. Gao, B. J. Lester, Y. Zhang, C. Wang, S. Rosenblum, L. Frunzio, L. Jiang, S. M. Girvin, and R. J. Schoelkopf, Programmable interference between two microwave quantum memories, Phys. Rev. X **8**, 021073 (2018).

[36] B. J. Chapman, S. J. de Graaf, S. H. Xue, Y. Zhang, J. Teoh, J. C. Curtis, T. Tsunoda, A. Eickbusch, A. P. Read, A. Koottandavida, S. O. Mundhada, L. Frunzio, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, High-on-off-ratio beam-splitter interaction for gates on bosonically encoded qubits, PRX Quantum **4**, 020355 (2023).

[37] J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, Science **352**, 325 (2016).

[38] D. von Lindenfels, O. Gräb, C. T. Schmiegelow, V. Kaushal, J. Schulz, M. T. Mitchison, J. Goold, F. Schmidt-Kaler, and U. G. Poschinger, Spin heat engine coupled to a harmonic-oscillator flywheel, Phys. Rev.

Lett. **123**, 080602 (2019).
[39] N. Van Horne, D. Yum, T. Dutta, P. Hänggi, J. Gong, D. Poletti, and M. Mukherjee, Single-atom energy-conversion device with a quantum load, npj Quantum Inf. **6**, 37 (2020).
[40] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O'Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Synthesizing arbitrary quantum states in a superconducting resonator, Nature **459**, 546 (2009).
[41] S. Krastanov, V. V. Albert, C. Shen, C.-L. Zou, R. W. Heeres, B. Vlastakis, R. J. Schoelkopf, and L. Jiang, Universal control of an oscillator with dispersive coupling to a qubit, Phys. Rev. A **92**, 040303(R) (2015).
[42] R. W. Heeres, B. Vlastakis, E. Holland, S. Krastanov, V. V. Albert, L. Frunzio, L. Jiang, and R. J. Schoelkopf, Cavity state manipulation using photon-number selective phase gates, Phys. Rev. Lett. **115**, 137002 (2015).
[43] R. W. Heeres, P. Reinhold, N. Ofek, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, Implementing a universal gate set on a logical qubit encoded in an oscillator, Nat. Commun. **8**, 94 (2017).
[44] Y. Y. Gao, B. J. Lester, K. S. Chou, L. Frunzio, M. H. Devoret, L. Jiang, S. M. Girvin, and R. J. Schoelkopf, Entanglement of bosonic modes through an engineered exchange interaction, Nature **566**, 509 (2019).