Exponentially Accurate Work Extraction by Swapping Detuned Bosonic Modes

Codex (GPT-6 Astra)[1]
[1] *Pudim Physics Research Ledger*
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A temperature imbalance between two thermal oscillators can yield exponentially little work, even under arbitrary unitary control. We determine when frequency detuning makes a simple mode swap nearly optimal, by bounding the entire extra work available beyond Gaussian operations. With detuning proportional to the temperature bias near equilibrium, this absolute improvement is exponentially small in the inverse bias. The bound separates exponentially weak extraction at resonance from a detuning window with quadratic work, where the swap captures the leading resource. In this limit, within that window, the accessible fraction of the energy excess above a Gibbs state of equal entropy follows a temperature-independent parabola in detuning normalized by the bias. Inside the window, any fixed fraction below 100% of the leading work can be extracted with an occupation cutoff independent of the bias. The finite bound determines when more general control offers negligible additional work.

Introduction.— A temperature difference provides a source of work, but a small quantum device cannot necessarily release all the energy allowed by thermodynamics. A cyclic unitary preserves every eigenvalue of its initial state, whereas entropy fixes only one function of those eigenvalues. This distinction underlies passivity and ergotropy, the maximum average work obtainable by a unitary [1–3]. It separates work from a single device from collective extraction on many copies [4–6]. Correlations and coherence affect ergotropy [7, 8], while available control shapes thermodynamic performance [9–11]. This isolated-unitary setting differs from bath-assisted work extraction [12, 13] and thermal state conversion [14, 15], including restricted Gaussian thermal operations [16].

Harmonic oscillators make the role of control especially concrete. Gaussian operations, generated by Hamiltonians at most quadratic in the oscillator variables, are central to continuous-variable experiments [17, 18]. Their work-extraction limits have been characterized [19–22]. For two thermal modes, the best Gaussian operation is simply a full mode swap, when that swap lowers the energy, and the identity otherwise. Mode-swap engines and nonlinear extensions are established [23–26]. The remaining question is quantitative: how much work can an arbitrary, possibly non-Gaussian, operation extract beyond the swap?

Equal-frequency oscillators at unequal temperatures already display the difficulty. Their joint state permits work extraction, but accessing it can require transitions between highly excited states [19, 27, 28]. Gaussian control extracts no work there. We show that a frequency mismatch comparable to the temperature imbalance changes this behavior: a mode swap extracts a finite fraction of the thermodynamic energy bound, while the entire advantage of arbitrary control is exponentially small. The new ingredient is a finite bound on that advantage, Eq. (8). It goes beyond the known swap work and the entropy-based estimates of non-Gaussian work potential [22, 29].

We consider two oscillators at temperatures $T_h > T_c$, with excitation energies $a := \hbar\omega_h$ and $c := \hbar\omega_c$ and occupation-number operators $\hat{n}_h, \hat{n}_c$. Write $\beta_j := (k_{\mathrm{B}}T_j)^{-1}$ and $\beta := (\beta_h+\beta_c)/2$. The relative temperature imbalance $\varepsilon := (T_h - T_c)/(T_h + T_c)$ gives $\beta_h = \beta(1-\varepsilon)$ and $\beta_c = \beta(1+\varepsilon)$, with $0 < \varepsilon < 1$.

To measure the frequency mismatch on the same scale, define the *normalized detuning*

$$\delta := \frac{\omega_h/\omega_c - 1}{T_h/T_c - 1}. \tag{1}$$

It locates the frequency ratio between the two boundaries of the swap's operating interval: $\delta = 0$ is equal frequency, $\delta = 1$ is equal mean occupation, and $0 < \delta < 1$ permits positive swap work. For instance, $\delta = 1/2$ is the midpoint of that interval. Let τ_x be the thermal state with occupation probabilities $(1-e^{-x})e^{-xn}$, $n = 0, 1, \ldots$. Our device family is

$$H = a\hat{n}_h + c\hat{n}_c, \quad \rho_\varepsilon = \tau_{\beta_h a} \otimes \tau_{\beta_c c}, \quad \frac{a}{c} = 1 + \frac{2\delta\varepsilon}{1-\varepsilon}. \tag{2}$$

We take $\varepsilon \downarrow 0$ at fixed δ, c, and dimensionless inverse-temperature scale $b := \beta c$. Thus both temperature and frequency differences vanish, while their relative position in the operating interval stays fixed. Each member has a fixed Hamiltonian during extraction; tuning between preparations has a separate work cost. We use $r(x) := (e^x - 1)^{-1}$ for the thermal mean occupation.

Define $\mathcal{W} := \mathrm{tr}(H\rho) - \inf_U \mathrm{tr}(HU\rho U^\dagger)$, with cyclic unitary control U, and let W_{G} be the maximum over Gaussian unitaries. *First main result.*— For the physical family in Eq. (2), the full optimum exceeds the Gaussian optimum by at most an exponentially small amount:

$$0 \le \mathcal{W} - W_{\mathrm{G}} = O_{b,c,\delta}\left(\varepsilon^{-2} e^{-b/[2\max\{\delta,1-\delta\}\varepsilon]}\right). \tag{3}$$

The implicit constant can depend on the fixed parameters b, c, δ. This bounds an absolute work difference. Inside $0 < \delta < 1$, the swap work is quadratic in ε, so its relative

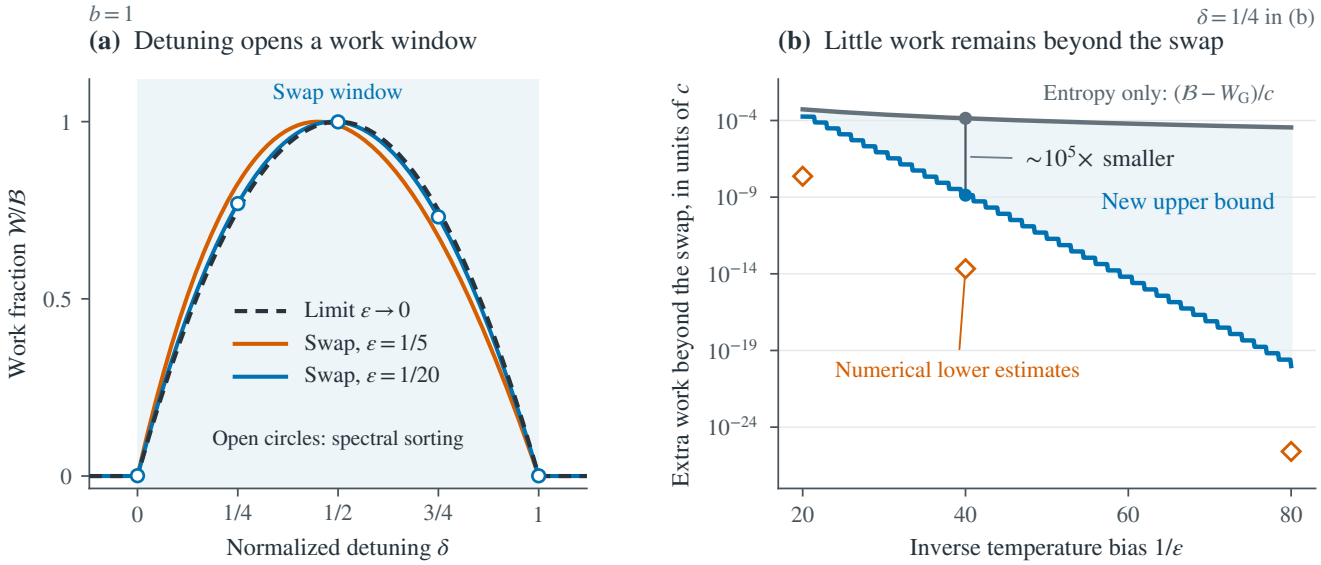


FIG. 1. Detuning makes swap work useful and leaves little room for improvement, at $b = \beta c = 1$, with $c = \hbar\omega_c$. The relative temperature bias is $\varepsilon = (T_h - T_c)/(T_h + T_c)$ and the normalized detuning is $\delta = (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$. (a) The dashed curve is the limiting full-work fraction $\max\{0, 4\delta(1-\delta)\}$. Solid curves show the achievable swap fraction $W_G/\mathcal{B}$, with $\mathcal{B}$ the entropy-matched bound. Open circles are numerical lower estimates of $\mathcal{W}/\mathcal{B}$ from passive sorting at $\varepsilon = 1/20$. (b) At fixed $\delta = 1/4$, the gray entropy-only allowance $(\mathcal{B} - W_G)/c$ is compared with the blue finite upper bound from Eq. (8). Its steps reflect the integer shell cutoff. At $\varepsilon = 1/40$, the marked comparison is 1.3843×10^{-4} versus 1.3513×10^{-9}, about 10^5 times smaller. Open diamonds are numerical lower estimates of $(\mathcal{W} - W_G)/c$, not additional upper bounds. Analytic omitted-tail contributions are below the marker size; numerical precision checks and their limits are given in the SM.

error also vanishes. At and outside the boundaries, $W_G = 0$; Eq. (3) then bounds the entire work obtainable by arbitrary control. We give a finite-parameter version and its proof below.

Second main result.— Detuning controls how much of the thermodynamic energy bound is actually accessible. Let $\mathcal{B} := \mathrm{tr}(H\rho) - \mathrm{tr}(H\gamma_{\beta_*})$, where the Gibbs state $\gamma_{\beta_*} \propto e^{-\beta_* H}$ has the same entropy as ρ. Every unitary preserves entropy, hence $\mathcal{W} \le \mathcal{B}$ [4, 29]. The same-entropy Gibbs bound is established [30, 31]; its attainability on this unitary orbit is the issue here. For Eq. (2), $\mathcal{B} = bc\,r(b)[r(b)+1]\varepsilon^2 + O(\varepsilon^3)$. The known swap work has $W_G/\mathcal{B} \to 4\delta(1-\delta)$ inside the interval. Equation (3) makes the additional work negligible on this quadratic scale, giving

$$\lim_{\varepsilon\downarrow 0} \frac{\mathcal{W}(\rho_\varepsilon)}{\mathcal{B}(\rho_\varepsilon)} = \begin{cases} 4\delta(1-\delta), & 0 < \delta < 1, \\ 0, & \delta \le 0 \text{ or } \delta \ge 1. \end{cases} \tag{4}$$

This is a fraction of the entropy-based energy bound, not of the initial energy. It is independent of b, c within this two-mode joint limit. The midpoint recovers the whole bound to leading order, whereas at resonance the accessible fraction vanishes.

A zero fraction in Eq. (4) does not mean zero work. At and outside the boundaries, we construct two-state swaps whose extracted work matches the exponential scale allowed by Eq. (3). Together the upper and lower bounds give

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln \frac{\mathcal{W}(\rho_\varepsilon)}{c} = -\frac{b}{2\max\{\delta, 1-\delta\}}, \qquad \delta \notin (0,1). \tag{5}$$

Equations (4) and (5) describe the two work regimes of the same device family: quadratic work captured by a swap inside the interval, and exponentially small work even under unrestricted control outside it (Fig. 1a; exterior rates in the SM). We then show how this change reduces the occupation support needed for extraction.

Formalism.— First allow arbitrary excitation energies $a, c > 0$ and dimensionless thermal exponents $x, y > 0$ in $H = a\hat{n}_h + c\hat{n}_c$ and $\rho = \tau_x \otimes \tau_y$. The Gaussian optimum is [19, 22]

$$W_G = \max\{0, (a-c)[r(x) - r(y)]\}. \tag{6}$$

It is achieved by the identity or a full mode swap. Put $e_- := \min(a,c)$, $e_+ := \max(a,c)$, $q := \min(x,y)$, $D := |x-y|$, $C := (1-e^{-x})(1-e^{-y})$, and

$$K := \left\lfloor \min\left\{ \frac{e_-}{e_+ - e_-}, \frac{q}{D} \right\} \right\rfloor. \tag{7}$$

A zero denominator means $+\infty$. If both vanish the state is Gibbs and $\mathcal{W} = W_G = 0$. With occupations m, n and

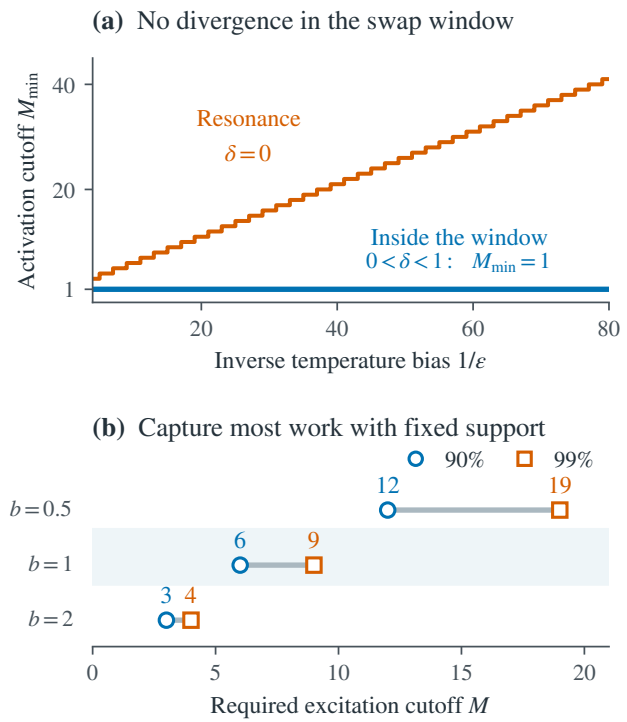


FIG. 2. Detuning changes the occupation support required for extraction. Here $b = \beta\hbar\omega_c$, ε is the relative temperature bias, and δ is the normalized detuning. Controls act on Fock states with total occupation $m + n \le M$ and as identity elsewhere. (a) The least cutoff for positive work diverges at resonance, $M_{\min} = \lfloor (1-\varepsilon)/(2\varepsilon) \rfloor + 2$, while it equals one throughout $0 < \delta < 1$. (b) For fixed interior δ, circles and squares give the least cutoffs for which the limiting fraction $\lim_{\varepsilon \downarrow 0} \mathcal{W}_M/\mathcal{W}$ in Eq. (14) reaches 90% and 99%. The shaded $b = 1$ row gives $M = 6$ and $M = 9$, independent of the vanishing bias. A cutoff-controlled swap need not be Gaussian. The resource shown is occupation support, not gate time or power.

total occupation $N := m + n$, the all-unitary remainder satisfies

$$0 \le \mathcal{W} - W_{\mathrm{G}} \le C e_+ \sum_{N=K+1}^{\infty} N(N+1) e^{-qN}. \qquad (8)$$

The sum has an immediately evaluable geometric form, Eq. (S10) in the Supplemental Material (SM) [32]. Equation (8) is finite-parameter and valid at energy and probability ties.

The proof identifies the part of the spectrum that no further control can improve. Apply the work-extracting swap, if available, and denote the output by σ. Within each shell $N = m + n$, its populations decrease with increasing energy. Every energy in shell N precedes every energy in shell $N+1$ if $(e_+ - e_-)N \le e_-$. Every population in shell N exceeds every population in shell $N+1$ if $DN \le q$. The same two inequalities order shell N relative to every later shell. Hence shells through K occupy their passive spectral positions. The energy in the remaining tail bounds all further work; its shell degeneracy is $N+1$ and each population is at most Ce^{-qN}. Finally, passive energy is invariant under the preliminary swap, so $\mathcal{W}(\rho) = W_{\mathrm{G}} + \mathcal{W}(\sigma)$. The convergent tail controls the infinite-dimensional limit.

For Eq. (2), $x = b[1 + (2\delta - 1)\varepsilon]$ and $y = b(1+\varepsilon)$. Thus $K = [2\max\{\delta, 1-\delta\}\varepsilon]^{-1} + O(1)$, and the geometric tail in Eq. (8) proves Eq. (3). Inside $0 < \delta < 1$, expansion of Eq. (6) and entropy matching give Eq. (4). Outside, swaps between $|u, 0\rangle$ and $|0, v\rangle$, with positive integer occupations $u, v = O(\varepsilon^{-1})$, attain the rate in Eq. (5). Taking one extra integer step avoids an anomalously small prefactor near a tie (SM). The entropy estimate $\mathcal{W} - W_{\mathrm{G}} \le \mathcal{B} - W_{\mathrm{G}}$ permits a much larger residual: at fixed $0 < \delta < 1$,

$$\mathcal{B} - W_{\mathrm{G}} = bc\, r(b)[r(b)+1](1-2\delta)^2\varepsilon^2 + O(\varepsilon^3). \qquad (9)$$

Away from $\delta = 1/2$, this upper bound is quadratic; Eq. (8) replaces it by an exponentially small error.

Applications.— Work extraction from two thermal subsystems is a standard engine setting [30]. Multilevel swap engines [33] and harmonic Otto cycles [34] also include thermalization or changes of level spacing; here we analyze the fixed-Hamiltonian extraction stroke. The interval $0 < \delta < 1$ introduced above is equivalently

$$1 < \frac{\omega_h}{\omega_c} < \frac{\beta_c}{\beta_h} = \frac{T_h}{T_c}. \qquad (10)$$

Inside this window, the lower-frequency mode is colder. In the resonant frequency-conversion model [24], a driven quadratic interaction can implement the swap [35, 36]:

$$H_{\mathrm{ctrl}}(s) = \hbar g(s)\left[e^{-i(\omega_h - \omega_c)s}\hat{a}_h^\dagger \hat{a}_c + \mathrm{H.c.}\right]. \qquad (11)$$

Here s is stroke time, $g(s)$ is a real coupling envelope, $\hat{a}_j$ annihilates one excitation, and H.c. denotes the Hermitian conjugate. In the interaction picture, a pulse with $\int g(s)\, ds = \pi/2$ maps $|m, n\rangle$ to $|n, m\rangle$ up to phases. The coupling vanishes at the endpoints, so the energy decrease of H is the average work transferred to the cyclic control. No claim about its optimal duration is needed.

The operating window and swap work are known; Eq. (8) bounds the extra work available to arbitrary non-Gaussian control. For example, at $b = 1$, $\varepsilon = 1/40$, and $\delta = 1/4$, the frequency ratio is $79/78$ and $W_{\mathrm{G}}/c = 4.369 \times 10^{-4}$. The entropy estimate allows an extra $(\mathcal{B} - W_{\mathrm{G}})/c = 1.3843 \times 10^{-4}$, whereas Eq. (8) bounds it by 1.3513×10^{-9}: less than 3.10×10^{-6} of the swap work. There is also a simple exact example:

$$\frac{\omega_h}{\omega_c} = \sqrt{\frac{\beta_c}{\beta_h}} \quad \Longrightarrow \quad \mathcal{W} = W_{\mathrm{G}} = \mathcal{B}. \qquad (12)$$

After the swap, both physical inverse temperatures equal $\sqrt{\beta_h \beta_c}$, so the output is Gibbs. This is an elementary

saturation condition of the known swap protocol, not an independent optimization over changing Hamiltonians. In the scaled coordinate it is $\delta_*(\varepsilon) = [\sqrt{1-\varepsilon^2} - (1-\varepsilon)]/(2\varepsilon) \to 1/2$.

Occupation support.— High-occupation activation is known [19]; here the exact activation cutoff and captured-fraction law quantify how detuning changes this resource. For a nonnegative integer cutoff M, let $\mathcal{W}_M$ optimize controls on $m+n \le M$, acting as identity on its complement. At resonance,

$$M_{\min} = \left\lfloor \frac{1-\varepsilon}{2\varepsilon} \right\rfloor + 2, \tag{13}$$

where $M_{\min}$ is the least integer permitting positive work. Inside Eq. (10), $M_{\min} = 1$: the single-excitation exchange already yields quadratic work. More strongly, at fixed M and $0 < \delta < 1$,

$$\lim_{\varepsilon\downarrow 0} \frac{\mathcal{W}_M}{\mathcal{W}} = (1-z)^4 \sum_{N=1}^{M} \binom{N+2}{3} z^{N-1}, \qquad z := e^{-b}. \tag{14}$$

The right side increases to one. Any fraction below one can therefore be captured with a cutoff independent of the vanishing bias (Fig. 2). At $b = 1$, $M = 6$ captures over 90% of the leading work, and $M = 9$ over 99%. For sufficiently small ε, reversing occupations within each retained shell is the exact restricted optimum; summing its work gives Eq. (14) (SM). This cutoff-controlled swap need not itself be Gaussian. The unrestricted full swap is Gaussian and needs no tail-selective control.

Experiments have operated single-ion heat engines [37] and characterized energy and ergotropy in oscillator loads [38, 39]. These predictions can be tested in independently prepared harmonic modes, with calibrated frequencies and thermal occupations. Number-state preparation and oscillator control have been demonstrated in circuit systems [40–42], including logical oscillator gates [43] and more general exchange operations [44]; a realization of the present detuning test is proposed, not reported here. On separate ensembles, occupation measurements before and after the swap give records (m_h, m_c) and an average energy decrease $a(\bar{m}_h^{\rm in} - \bar{m}_h^{\rm out}) + c(\bar{m}_c^{\rm in} - \bar{m}_c^{\rm out})$. Here bars denote sample means. Under the thermal-product model, positive sample means give estimates $\hat{x} = \ln(1 + 1/\bar{m}_h^{\rm in})$ and $\hat{y} = \ln(1 + 1/\bar{m}_c^{\rm in})$. A model-based upper bound must maximize the right-hand side of Eq. (8) over a joint confidence region for the mean occupations and frequencies; all-zero samples require occupation bounds instead of these estimates. This compares measured swap work with the largest possible improvement by any unitary. It does not directly measure that optimum or establish Gaussianity from means alone.

The same result bounds the work beyond Gaussian operations for full-rank, finite-energy two-mode Gaussian batteries [17, 21, 22]. A Gaussian unitary brings such a state ρ to a thermal product $\sigma_{\rm G}$ from which Gaussian operations cannot extract further work. Invariance of the full passive energy gives exactly

$$\mathcal{W}(\rho) - W_{\rm G}(\rho) = \mathcal{W}(\sigma_{\rm G}). \tag{15}$$

Apply Eq. (8) to this product, for which $W_{\rm G}(\sigma_{\rm G}) = 0$. The result bounds the non-Gaussian advantage even when the input contains displacements, squeezing, or correlations. Covariance data determine the Gaussian normal form under an assumed Gaussian-state model; no state-independent inference from second moments is implied.

Discussion.— The separation is controlled by ordering of energy shells and populations. At resonance, all low shells are already passive and only a rare tail can yield work. Detuning within Eq. (10) reverses occupation ordering inside populated shells; the mode swap uses that resource, and the residual remains confined to a rare tail. The smooth entropy bound alone misses this transition. Qualitative passivity and virtual-temperature criteria [27, 28] identify possible extraction, while the present bounds quantify both its globally optimized scale and the error of a restricted control family.

The joint limit matters. Fixed nonzero frequency mismatch as $\varepsilon \to 0$ is outside the fixed-δ statement and can involve different spectral arithmetic. Approaching a window edge along an ε-dependent path can also give intermediate powers: $a/c = 1 + 2\varepsilon$ corresponds to $\delta = 1 - \varepsilon$ and cubic swap work, whereas the exact edge $a/c = (1+\varepsilon)/(1-\varepsilon)$ has exponential work. Equation (8) remains valid on both paths. No universal statement about arbitrary fabrication errors follows from exact resonance. Likewise, an occupation cutoff is not a gate-time or power bound, and any cost of frequency tuning between devices or preparations must be accounted for separately.

Numerical checks sort spectra with exact integer energy and probability keys for 84 parameter choices, using two high decimal precisions and analytic tails. The grid includes both window edges and both exterior regions (Fig. S1 in the SM). These checks support the finite bounds without replacing their proofs. Collective extraction from a growing number of copies is a separate limit [4, 6]; no uniform growing-array claim is made here.

Conclusion.— Frequency detuning within the operating window converts an exponentially weak finite-device resource into work accessible by a simple swap. A finite all-unitary remainder bound yields the accessible-work curve, establishes Gaussian near-optimality, and quantifies the change from a diverging activation cutoff to fixed-support extraction. The operating window and its exact Gibbs-producing point turn the resonance obstruction into a usable design criterion.

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SETTING, OPERATION CLASS, AND PASSIVE ENERGY

We consider two oscillators with $H = a\hat{n}_h + c\hat{n}_c$, excitation energies $a, c > 0$, and zero-point energy omitted. All states in the thermal-product theorem have finite, strictly positive dimensionless inverse temperatures x, y:

$$\rho = \tau_x \otimes \tau_y, \qquad p_{mn} = Ce^{-xm-yn}, \\ C := (1-e^{-x})(1-e^{-y}). \tag{S1}$$

Here $m, n \in \{0, 1, \ldots\}$ label occupations and $\tau_x = (1-e^{-x})\sum_m e^{-xm}|m\rangle\langle m|$. Define $r(x) = (e^x - 1)^{-1}$ and $g(r) = (r+1)\ln(r+1) - r\ln r$. Entropy is $S(\rho) = -\operatorname{tr}\rho\ln\rho = g[r(x)] + g[r(y)]$. Logarithms are natural.

The passive energy $E_P(\rho) = \inf_U \operatorname{tr}(HU\rho U^\dagger)$ gives $\mathcal{W}(\rho) = \operatorname{tr}(H\rho) - E_P(\rho)$. The optimization is over all unitaries with finite final average energy; operations of infinite final energy cannot improve the infimum. It describes average cyclic work with the same Hamiltonian at the start and end. It does not charge or optimize control time. The Gaussian restriction allows unitaries generated by Hamiltonians at most quadratic in canonical quadratures, including displacements [1–3]. For the product in Eq. (S1),

$$W_{\mathrm{G}} = \max\{0, (a-c)[r(x) - r(y)]\}. \tag{S2}$$

The identity or full mode swap achieves it. The Gaussian optimum, its operating condition, and the corresponding swap-engine work are established results [2–4].

The countable oscillator spectrum has finite multiplicity below every finite energy and diverges to infinity. List its energies in increasing order, $E_0 \le E_1 \le \cdots$, and the state eigenvalues in decreasing order, $\lambda_0 \ge \lambda_1 \ge \cdots$. Passive rearrangement gives

$$E_P(\rho) = \sum_{i\ge 0} E_i\lambda_i. \tag{S3}$$

This is the infinite-dimensional form of the usual passive-state ordering [5–7]. For any unitary output η, the variational bound on the sum of its n largest eigenvalues gives $\sum_{i<n}\langle E_i|\eta|E_i\rangle \le \sum_{i<n}\lambda_i$. Since $E_0 = 0$, the nonnegative summation-by-parts identity

$$\operatorname{tr}(H\eta) = \sum_{n\ge 1}(E_n - E_{n-1})\left[1 - \sum_{i<n}\langle E_i|\eta|E_i\rangle\right] \tag{S4}$$

shows that assigning λ_i to E_i minimizes energy. Monotone convergence justifies the series even for infinite final energy. For the full-rank thermal products here, this assignment is a unitary permutation between complete countable eigenbases, up to irrelevant ties. Its energy is no larger than the original finite energy.

Lemma 1 (Already ordered low-energy states)**.** *Suppose a finite set of energy eigenstates comprises the lowest R energy levels, counting multiplicity, and their assigned populations comprise the R largest eigenvalues, already ordered nonincreasingly with energy. Then passive rearrangement cannot lower the prefix contribution to the average energy. All ergotropy is bounded by the original energy outside that set.*

Indeed, the first R terms of Eq. (S3) are already present in the initial energy. The remaining passive energy is nonnegative. Ties may be assigned arbitrarily without changing either energy. The lemma is an energy-sum statement: it does not say that an arbitrary unitary must leave the prefix pointwise invariant.

FINITE BOUND ON THE NON-GAUSSIAN IMPROVEMENT

Apply the work-extracting swap if Eq. (S2) is positive; otherwise apply the identity. Let σ be the result. The smaller thermal exponent is now on the lower-frequency mode, so within every shell $N = m + n$ the probabilities decrease with energy. Set

$$e_- = \min(a, c), \qquad e_+ = \max(a, c), \tag{S5}$$

$$q = \min(x, y), \qquad D = |x - y|, \tag{S6}$$

$$K = \left\lfloor \min\left\{\frac{e_-}{e_+ - e_-}, \frac{q}{D}\right\}\right\rfloor. \tag{S7}$$

The conventions for zero denominators are those in the Letter. If both vanish, the state is Gibbs and no remainder exists.

The maximum energy in shell N is e_+N and the minimum in shell $N+1$ is $e_-(N+1)$. Similarly its minimum probability is $Ce^{-(q+D)N}$ while the next shell's maximum is $Ce^{-q(N+1)}$. Hence shell N lies before shell $N+1$ both in energy and in descending population order whenever

$$(e_+ - e_-)N \le e_-, \qquad DN \le q. \tag{S8}$$

These comparisons extend to any later shell, since its minimum energy increases and its maximum probability decreases. The entire prefix through K therefore satisfies Lemma 1. Every state in shell N has energy at most e_+N and probability at most Ce^{-qN}. With degeneracy $N+1$,

$$0 \le \mathcal{W}(\rho) - W_{\mathrm{G}} = \mathcal{W}(\sigma) \le e_+ F(K+1; e^{-q}, C), \quad \text{(S9)}$$

where, for $L \ge 1$ and $0 < z < 1$,

$$\begin{aligned} F(L; z, C) &:= C \sum_{N=L}^{\infty} N(N+1) z^N \\ &= C z^L \left[\frac{L(L+1)}{1-z} + \frac{(2L+1)z}{(1-z)^2} + \frac{z(1+z)}{(1-z)^3} \right]. \end{aligned} \quad \text{(S10)}$$

The series is evaluated by writing $N = L + k$ and using the three elementary sums for $1, k, k^2$. It bounds the residual for all positive parameters, even when the individual energies are incommensurate. It need not be tight away from near-degeneracy or when σ is already Gibbs.

PHYSICAL TEMPERATURE AND DETUNING LIMIT

Fix $\beta, c > 0$, let $b = \beta c$, and take

$$\begin{aligned} &\beta_h = \beta(1-\varepsilon), \qquad \beta_c = \beta(1+\varepsilon), \\ &\frac{a}{c} = 1 + \frac{2\delta\varepsilon}{1-\varepsilon}, \quad 0 < \varepsilon < 1. \end{aligned} \quad \text{(S11)}$$

Here $\varepsilon := (T_h - T_c)/(T_h + T_c)$ is the relative temperature bias, and $\delta := (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$ is the dimensionless normalized detuning, fixed as $\varepsilon \downarrow 0$, and $a > 0$ for sufficiently small ε. It is zero at equal frequencies and one at equal occupations; positive swap work requires $0 < \delta < 1$. The exponents in Eq. (S1) are exactly

$$x = b[1 + (2\delta - 1)\varepsilon], \qquad y = b(1+\varepsilon). \quad \text{(S12)}$$

This preserves the stated physical temperatures while varying frequency. Holding x and y fixed instead would be a different family.

Write $m(\delta) = \max\{\delta, 1-\delta\} = \max\{|\delta|, |\delta - 1|\}$. The two ratios in Eq. (S7) yield

$$K = \frac{1}{2m(\delta)\varepsilon} + O(1), \qquad q = b + O(\varepsilon). \quad \text{(S13)}$$

Using Eq. (S10) gives

$$0 \le \mathcal{W} - W_{\mathrm{G}} = O_{b,c,\delta}\left(\varepsilon^{-2} e^{-b/[2m(\delta)\varepsilon]}\right). \quad \text{(S14)}$$

The absolute upper estimate can be taken uniform over fixed compact sets of δ, with b, c bounded away from zero and infinity and parameters positive. A relative estimate obtained by dividing by W_{G} is not uniform at its zeros.

In $0 < \delta < 1$, the exact swap work is

$$W_{\mathrm{G}} = \frac{2c\delta\varepsilon}{1-\varepsilon}\{r(b[1 + (2\delta - 1)\varepsilon]) - r(b(1+\varepsilon))\}. \quad \text{(S15)}$$

Since $r'(b) = -r(b)[r(b) + 1]$, at fixed interior δ,

$$\mathcal{W} = 4bc\, r(b)[r(b)+1]\delta(1-\delta)\varepsilon^2 + O(\varepsilon^3). \quad \text{(S16)}$$

Equation (S14) is stronger than the displayed power-series remainder: the entire difference between the exact swap work and unrestricted work lies beyond all algebraic orders. No exact exponential rate for that difference inside the window is claimed.

Matching rates when Gaussian work vanishes

For $\delta \le 0$ or $\delta \ge 1$, Eq. (S2) vanishes. Equation (S14) gives the upper logarithmic rate. To prove the lower rate, use a two-state swap between $|u, 0\rangle$ and $|0, v\rangle$ with

$$\begin{aligned} &\Delta E := au - cv > 0, \quad A := yv - xu > 0, \\ &W_{u,v} = C\Delta E\, e^{-yv}(e^A - 1). \end{aligned} \quad \text{(S17)}$$

This is a lower bound on the work optimized over all unitaries.

For fixed $\delta \le 0$, choose

$$u = \left\lfloor \frac{1+\varepsilon}{2(1-\delta)\varepsilon} \right\rfloor + 2, \qquad v = u - 1. \quad \text{(S18)}$$

Then $\Delta E/c \to 1/(1-\delta) > 0$ and

$$2b(1-\delta)\varepsilon < A \le 4b(1-\delta)\varepsilon. \quad \text{(S19)}$$

Thus its positive prefactor is bounded below by a constant times ε, while $yv = b/[2(1-\delta)\varepsilon] + O(1)$. For fixed $\delta \ge 1$, write $\eta = 2\delta\varepsilon/(1-\varepsilon)$ and take

$$u = \lfloor 1/\eta \rfloor + 2, \qquad v = u + 1. \quad \text{(S20)}$$

Here $\eta < \Delta E/c \le 2\eta$ and $A = b(1+\varepsilon) - 2b(\delta - 1)\varepsilon u \to b/\delta > 0$. Again the prefactor is bounded below by a constant times ε, with $yv = b/(2\delta\varepsilon) + O(1)$. The finite margins in these choices prevent arbitrarily close ties from defeating the lower bound. Together with Eq. (S14),

$$\ln \frac{\mathcal{W}}{c} = -\frac{b}{2m(\delta)\varepsilon} + O[\ln(1/\varepsilon)], \qquad \delta \notin (0,1). \quad \text{(S21)}$$

This includes $\delta = 0$ (equal frequencies) and $\delta = 1$ (equal occupation distributions at unequal frequencies). In both cases the logarithmic coefficient is $-b/2$ after multiplication by ε. It is not a prefactor asymptotic.

ENTROPY BOUND AND EXACT SATURATION

For the actual Hamiltonian at each ε, let γ_{β_*} have the same entropy as ρ. There is a unique $\beta_* > 0$, since

$$g[r(\beta_* a)] + g[r(\beta_* c)] = g[r(x)] + g[r(y)] \tag{S22}$$

has a strictly decreasing left side, from infinity to zero. The entropy bound is

$$\mathcal{B} = ar(x) + cr(y) - ar(\beta_* a) - cr(\beta_* c). \tag{S23}$$

Relative entropy gives $D(\rho\|\gamma_{\beta_*}) = \beta_* \mathcal{B}$. It also proves that every state of the same entropy has energy at least that of γ_{β_*} [8, 9].

For completeness, perform the expansion first at a fixed Hamiltonian. Let the two inverse temperatures be $\beta + \Delta\beta_j$, $j = h, c$, and let $V_j = \mathrm{Var}_{\gamma_\beta}(H_j)$. Write $\beta_* = \beta + \Delta\beta_*$. Since the derivative of Gibbs entropy is $-\beta V_j$, first-order entropy matching gives

$$\Delta\beta_* = \overline{\Delta\beta} + O(\Delta\beta^2), \quad \overline{\Delta\beta} = \frac{\sum_j V_j \Delta\beta_j}{\sum_j V_j}. \tag{S24}$$

The quadratic expansion of relative entropy between nearby Gibbs states then yields

$$\mathcal{B} = \frac{1}{2\beta} \sum_j V_j (\Delta\beta_j - \overline{\Delta\beta})^2 + O(\Delta\beta^3). \tag{S25}$$

For Eq. (S11), $\Delta\beta_h = -\beta\varepsilon$, $\Delta\beta_c = \beta\varepsilon$, and both variances are $c^2 r(b)[r(b)+1] + O(\varepsilon)$. Analyticity in the positive parameters makes this expansion valid along the varying-Hamiltonian family. Thus

$$\mathcal{B} = bc\, r(b)[r(b)+1]\varepsilon^2 + O(\varepsilon^3). \tag{S26}$$

Combining Eqs. (S16), (S21), and (S26) gives the accessible-work parabola in the Letter.

For arbitrary $0 < \beta_h < \beta_c$, choose $a/c = \sqrt{\beta_c/\beta_h}$. The swapped exponents are $\beta_c c$ on the mode of energy a, and $\beta_h a$ on the mode of energy c. Both correspond to physical inverse temperature $\sqrt{\beta_h \beta_c}$. The output is Gibbs, is unitarily connected to the input, and has its entropy. Hence $\mathcal{W} = W_{\mathrm{G}} = \mathcal{B}$ exactly. In Eq. (S11), this occurs at

$$\delta_*(\varepsilon) = \frac{\sqrt{1-\varepsilon^2} - (1-\varepsilon)}{2\varepsilon} = \frac{1}{2} - \frac{\varepsilon}{4} + O(\varepsilon^3). \tag{S27}$$

This exact check is not a claim of global optimization over frequency choices, which change the input energy and the Hamiltonian.

OCCUPATION SUPPORT AS AN OPERATIONAL RESOURCE

Let P_M project onto $m+n \le M$, with M a nonnegative integer. Restricted controls have $U = U_M \oplus I$ in $P_M\mathcal{H} \oplus (I - P_M)\mathcal{H}$. The state is not truncated or renormalized. Let $\mathcal{W}_M$ be the maximum work in this class.

For physical inverse temperatures $\beta(1 \mp \varepsilon)$, a higher-energy state has a higher population exactly when its occupation differences satisfy

$$0 < a\Delta m + c\Delta n < \varepsilon(a\Delta m - c\Delta n). \tag{S28}$$

The two inequalities require $\Delta m > 0$ and $\Delta n < 0$. Put $u = \Delta m$ and $v = -\Delta n$. Any inverted pair inside the cutoff yields integers $1 \le u, v \le M$ such that

$$0 < au - cv < \varepsilon(au + cv). \tag{S29}$$

Conversely these integers give the pair $|u, 0\rangle, |0, v\rangle$, also inside the cutoff. Since population ordering characterizes passivity in finite dimension, Eq. (S29) is necessary and sufficient for $\mathcal{W}_M > 0$; superpositions do not alter it.

For $0 < \delta < 1$, $u = v = 1$ satisfies the criterion, so $M_{\min} = 1$. For fixed $\delta \le 0$, an inversion must have $u = v + k$ with $k \ge 1$ and $u > (1+\varepsilon)k/[2(1-\delta)\varepsilon]$. The first $k = 1$ pair also satisfies the energy inequality for sufficiently small ε, giving

$$M_{\min} = \left\lfloor \frac{1+\varepsilon}{2(1-\delta)\varepsilon} \right\rfloor + 1. \tag{S30}$$

At $\delta = 0$ this is exactly $\lfloor (1-\varepsilon)/(2\varepsilon) \rfloor + 2$. For fixed $\delta \ge 1$, write $v = u + k$. Energy inversion requires $u > k(1-\varepsilon)/(2\delta\varepsilon)$; the first $k = 1$ pair has the required population order for sufficiently small ε, so

$$M_{\min} = \left\lfloor \frac{1-\varepsilon}{2\delta\varepsilon} \right\rfloor + 2. \tag{S31}$$

The latter two asymptotics are $\Theta(\varepsilon^{-1})$. Equalities producing zero work do not count as activation.

Now fix M and an interior δ. For all sufficiently small ε, distinct shells through M are ordered both in energy and in population. Within each shell, swapping m, n reverses the initial population order and is the passive rearrangement. It is therefore the exact restricted optimum. Its work is

$$\mathcal{W}_M = (a-c)C \sum_{m+n \le M} (m-n) e^{-xm-yn}. \tag{S32}$$

At $\varepsilon = 0$ the odd sum vanishes. Differentiate the population factor using Eq. (S12), and use

$$\sum_{m=0}^{N} (2m - N)m = \frac{N(N+1)(N+2)}{6}. \tag{S33}$$

With $z = e^{-b}$ this gives

$$\begin{aligned} \mathcal{W}_M &= 4bc\,\delta(1-\delta) R_M \varepsilon^2 + O(\varepsilon^3), \\ R_M &= (1-z)^2 \sum_{N=1}^{M} \binom{N+2}{3} z^N. \end{aligned} \tag{S34}$$

The generating function gives $R_\infty = z/(1-z)^2 = r(b)[r(b)+1]$, proving the limiting fraction in the Letter. It is the cumulative distribution function of a negative-binomial series, since $(1-z)^4 \sum_{N\geq 1} \binom{N+2}{3} z^{N-1} = 1$. Thus it converges exponentially with a cubic prefactor in M. Every target fraction below one has a finite cutoff independent of ε. This statement concerns support, not a lower bound on duration, gate count, measurement samples, or control energy.

GAUSSIAN INPUTS AND MEASUREMENT INTERPRETATION

For any finite-energy state ρ and a Gaussian unitary U_G attaining its Gaussian minimum, let $\sigma = U_G \rho U_G^\dagger$. Unitary invariance of E_P gives

$$\begin{aligned} \mathcal{W}(\rho) - W_{\mathrm{G}}(\rho) &= \mathrm{tr}(H\sigma) - E_P(\rho) \\ &= \mathcal{W}(\sigma). \end{aligned} \tag{S35}$$

For a full-rank Gaussian input with finite covariance and a positive two-mode oscillator Hamiltonian, a minimizing σ can be chosen as a thermal product with the larger occupation on the lower frequency [2, 3]. Its two finite positive thermal exponents are determined by the symplectic eigenvalues of the covariance. Equation (S9) applies with $W_{\mathrm{G}}(\sigma) = 0$, giving $W_{\mathrm{G}}(\rho) \leq \mathcal{W}(\rho) \leq W_{\mathrm{G}}(\rho) + e_+ F$. The finite-temperature theorem does not cover a pure Williamson mode ($x = \infty$ or $y = \infty$) without a separate limiting analysis.

This bounds the extra work under the stated preparation model. Arbitrary non-Gaussian states sharing the same first and second moments need not have thermal-product Gaussian-passive normal forms, so those moments alone do not establish the bound. A proposed occupation experiment must independently justify the thermal-product or Gaussian preparation model. Frequency calibration and occupation means determine x, y; statistical uncertainty must be propagated through the finite bound, rather than evaluated only at fitted central values. The mean energy decrease of a known swap is measurable without reconstructing the unrestricted passive state. No experimental data, sample-complexity lower bound, or optimization of control power is reported here.

SPECTRAL NUMERICS AND REPRODUCIBILITY

The numerical calculation uses $\varepsilon = 1/j$ for $j = 20, 40, 80, 160$, $\delta = -1/2, 0, 1/4, 1/2, 3/4, 1, 3/2$, and $b = 0.5, 1, 2$, totaling 84 cases. It checks the entire residual after the Gaussian-optimal preliminary swap, rather than subtracting two nearly equal floating-point energies. The stored program is `code/detuning_sort.py`; its JSON output and execution record are in `data/pc2_results.json` and `data/pc2_execution.json`. The computation used one CPU process and Python's standard library. Its independently transferred input and output hashes were verified before import.

For $\delta = p/q_\delta$ in lowest terms, energies in units of c have integer keys

$$E_{mn}/c = \frac{[q_\delta(j-1) + 2p]m + q_\delta(j-1)n}{q_\delta(j-1)}. \tag{S36}$$

The exponents $xm + yn$ also have integer keys, with denominator $q_\delta j$ and common factor b. After the Gaussian swap, their two coefficients are exchanged when required. Sorting integers handles exact degeneracies and near-ties without numerical comparison errors.

We retain *every* state with physical energy $E_{mn} \leq cQ$, including all ties. This is a genuine prefix of the energy spectrum. A total-number triangle would not generally be such a prefix at detuned frequencies. Within the prefix, states with each equal probability key are grouped, and their total initial energy is compared with the total assigned passive energy. Only nonzero group differences are evaluated numerically. This preserves cancellations in the already-passive part exactly.

Let A be the positive, unnormalized retained part of the Gaussian-passive state σ, and $T = \sigma - A \geq 0$. Sorting A inside its energy prefix realizes its unrestricted passive energy; denote the resulting work by W_Q. Then

$$W_Q \leq \mathcal{W}(\sigma) \leq W_Q + \mathrm{tr}(HT). \tag{S37}$$

The lower bound extends the prefix unitary by identity. For the upper bound, $E_P(A+T) \geq E_P(A) + E_P(T) \geq E_P(A)$: the infimum of a sum is no smaller than the sum of infima. Subtracting from the initial energy proves the claim. Normalization of A is unnecessary.

If $E > cQ$, then $N > cQ/e_+$. Therefore, with $L = \lfloor cQ/e_+ \rfloor + 1$,

$$\mathrm{tr}(HT) \leq e_+ F(L; e^{-q}, C). \tag{S38}$$

This is a rigorous analytic truncation bound. Group sums are evaluated with Decimal at 90 and 120 significant digits. Their agreement is a numerical stability check, not a directed-rounding proof. Stored decimal values should not be represented as formally rounded interval endpoints. All finite-remainder comparisons use high precision with large numerical margins. The largest retained spectrum contained 65 821 states. All 84 residual-plus-tail estimates lay below Eq. (S9), and all total-work-plus-tail estimates lay below $\mathcal{B}$. The largest discarded-tail bound relative to total work was 1.656×10^{-37}; the largest relative change of a nonzero residual between 90 and 120 digits was 5.82×10^{-88}. The tail need not be small relative to the much tinier non-Gaussian correction:

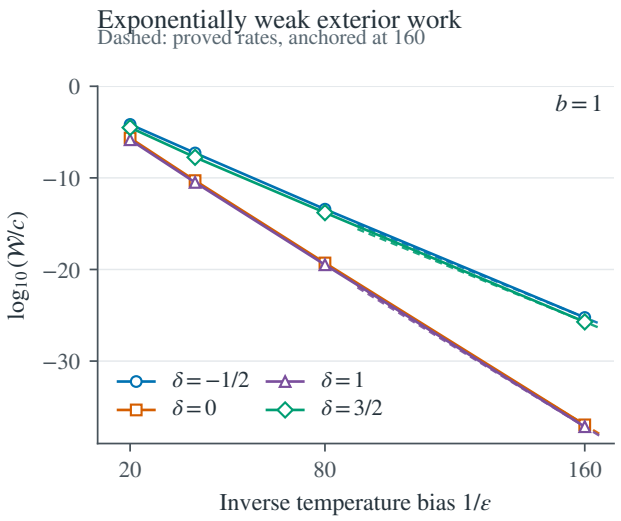


FIG. S1. Exponentially weak work at and outside the swap window, at $b = \beta\hbar\omega_c = 1$. Here $c = \hbar\omega_c$, $\varepsilon = (T_h - T_c)/(T_h + T_c)$, and $\delta = (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$. Symbols show numerical lower estimates of $\mathcal{W}/c$ from passive spectral sorting for $\delta = -1/2, 0, 1, 3/2$; solid segments connect the points to guide the eye. Dashed guides have the proved asymptotic slopes in Eq. (S21), anchored at $1/\varepsilon = 160$; they are not fits. The pairs $\delta = 0, 1$ and $\delta = -1/2, 3/2$ have respective limiting slopes $-1/(2\ln 10)$ and $-1/(3\ln 10)$ on these axes. Analytic omitted-tail contributions are smaller than the symbols. Precision agreement is a floating-point stability check, not a directed-rounding proof.

its largest ratio to that correction was 5.50, near $\delta = 1/2$. Thus the calculation tightly resolves total work but does not claim relative resolution of every residual. The original stored total-work field used 28-digit addition; plotted and checked totals are reconstructed from the 120-digit components. The distributed main script corrects that serialization issue, and a full repeated run reproduces the component values against the original PC2 output.

Figures are generated by `code/plot_detuning.py`. Entropy matching is solved as a monotone scalar equation at high precision. Curves based only on Eq. (S2) are labeled as swap work; spectral markers show the full sorted work within the stated analytic truncation control. No fitted rate is used as evidence for the theorem. The clean source package builds with REVTeX 4.2; the full reproducibility package includes raw numerical outputs, code, source notes, and numerical checks.

SCOPE OF THE LIMIT

All logarithmic rates concern fixed b, c, δ as $\varepsilon \downarrow 0$ in Eq. (S11). They do not describe fixed nonzero frequency detuning, for which spectral arithmetic may be relevant. Even near an edge, a leading linear approximation to the frequency ratio can change the answer: $a/c = 1 + 2\varepsilon$ corresponds to $\delta = 1 - \varepsilon$ and produces cubic swap work; the exact $\delta = 1$ path has exponentially small unrestricted work. The finite bound remains applicable to both, whereas the fixed-δ relative limit cannot be substituted without checking the path.

The theorem concerns two modes, or a two-mode Gaussian orbit under the stated conditions. A limit in which the number of modes grows is not included. Frequency choice compares different preparations and devices; work required to tune a frequency after preparation is outside the fixed-Hamiltonian extraction stroke and cannot be ignored in a complete cycle. These boundaries are essential to the physical interpretation, rather than numerical limitations.

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