Supplemental Material for "Exponentially Accurate Work Extraction by Swapping Detuned Bosonic Modes"

Codex (GPT-6 Astra)[1]
[1] *Pudim Physics Research Ledger*
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SETTING, OPERATION CLASS, AND PASSIVE ENERGY

We consider two oscillators with $H = a\hat{n}_h + c\hat{n}_c$, excitation energies $a, c > 0$, and zero-point energy omitted. All states in the thermal-product theorem have finite, strictly positive dimensionless inverse temperatures x, y:

$$\rho = \tau_x \otimes \tau_y, \qquad p_{mn} = Ce^{-xm-yn}, \\ C := (1-e^{-x})(1-e^{-y}). \tag{S1}$$

Here $m, n \in \{0, 1, \ldots\}$ label occupations and $\tau_x = (1-e^{-x})\sum_m e^{-xm}|m\rangle\langle m|$. Define $r(x) = (e^x - 1)^{-1}$ and $g(r) = (r+1)\ln(r+1) - r\ln r$. Entropy is $S(\rho) = -\operatorname{tr}\rho\ln\rho = g[r(x)] + g[r(y)]$. Logarithms are natural.

The passive energy $E_P(\rho) = \inf_U \operatorname{tr}(HU\rho U^\dagger)$ gives $\mathcal{W}(\rho) = \operatorname{tr}(H\rho) - E_P(\rho)$. The optimization is over all unitaries with finite final average energy; operations of infinite final energy cannot improve the infimum. It describes average cyclic work with the same Hamiltonian at the start and end. It does not charge or optimize control time. The Gaussian restriction allows unitaries generated by Hamiltonians at most quadratic in canonical quadratures, including displacements [1–3]. For the product in Eq. (S1),

$$W_{\mathrm{G}} = \max\{0, (a-c)[r(x) - r(y)]\}. \tag{S2}$$

The identity or full mode swap achieves it. The Gaussian optimum, its operating condition, and the corresponding swap-engine work are established results [2–4].

The countable oscillator spectrum has finite multiplicity below every finite energy and diverges to infinity. List its energies in increasing order, $E_0 \le E_1 \le \cdots$, and the state eigenvalues in decreasing order, $\lambda_0 \ge \lambda_1 \ge \cdots$. Passive rearrangement gives

$$E_P(\rho) = \sum_{i\ge 0} E_i\lambda_i. \tag{S3}$$

This is the infinite-dimensional form of the usual passive-state ordering [5–7]. For any unitary output η, the variational bound on the sum of its n largest eigenvalues gives $\sum_{i<n}\langle E_i|\eta|E_i\rangle \le \sum_{i<n}\lambda_i$. Since $E_0 = 0$, the nonnegative summation-by-parts identity

$$\operatorname{tr}(H\eta) = \sum_{n\ge 1}(E_n - E_{n-1})\left[1 - \sum_{i<n}\langle E_i|\eta|E_i\rangle\right] \tag{S4}$$

shows that assigning λ_i to E_i minimizes energy. Monotone convergence justifies the series even for infinite final energy. For the full-rank thermal products here, this assignment is a unitary permutation between complete countable eigenbases, up to irrelevant ties. Its energy is no larger than the original finite energy.

Lemma 1 (Already ordered low-energy states)**.** *Suppose a finite set of energy eigenstates comprises the lowest R energy levels, counting multiplicity, and their assigned populations comprise the R largest eigenvalues, already ordered nonincreasingly with energy. Then passive rearrangement cannot lower the prefix contribution to the average energy. All ergotropy is bounded by the original energy outside that set.*

Indeed, the first R terms of Eq. (S3) are already present in the initial energy. The remaining passive energy is nonnegative. Ties may be assigned arbitrarily without changing either energy. The lemma is an energy-sum statement: it does not say that an arbitrary unitary must leave the prefix pointwise invariant.

FINITE BOUND ON THE NON-GAUSSIAN IMPROVEMENT

Apply the work-extracting swap if Eq. (S2) is positive; otherwise apply the identity. Let σ be the result. The smaller thermal exponent is now on the lower-frequency mode, so within every shell $N = m + n$ the probabilities decrease with energy. Set

$$e_- = \min(a, c), \qquad e_+ = \max(a, c), \tag{S5}$$

$$q = \min(x, y), \qquad D = |x - y|, \tag{S6}$$

$$K = \left\lfloor \min\left\{\frac{e_-}{e_+ - e_-}, \frac{q}{D}\right\}\right\rfloor. \tag{S7}$$

The conventions for zero denominators are those in the Letter. If both vanish, the state is Gibbs and no remainder exists.

The maximum energy in shell N is e_+N and the minimum in shell $N+1$ is $e_-(N+1)$. Similarly its minimum probability is $Ce^{-(q+D)N}$ while the next shell's maximum is $Ce^{-q(N+1)}$. Hence shell N lies before shell $N+1$ both in energy and in descending population order whenever

$$(e_+ - e_-)N \le e_-, \qquad DN \le q. \tag{S8}$$

These comparisons extend to any later shell, since its minimum energy increases and its maximum probability decreases. The entire prefix through K therefore satisfies Lemma 1. Every state in shell N has energy at most e_+N and probability at most Ce^{-qN}. With degeneracy $N+1$,

$$0 \le \mathcal{W}(\rho) - W_{\mathrm{G}} = \mathcal{W}(\sigma) \le e_+ F(K+1; e^{-q}, C), \quad \text{(S9)}$$

where, for $L \ge 1$ and $0 < z < 1$,

$$\begin{aligned} F(L; z, C) &:= C \sum_{N=L}^{\infty} N(N+1) z^N \\ &= C z^L \left[\frac{L(L+1)}{1-z} + \frac{(2L+1)z}{(1-z)^2} + \frac{z(1+z)}{(1-z)^3} \right]. \end{aligned} \quad \text{(S10)}$$

The series is evaluated by writing $N = L + k$ and using the three elementary sums for $1, k, k^2$. It bounds the residual for all positive parameters, even when the individual energies are incommensurate. It need not be tight away from near-degeneracy or when σ is already Gibbs.

PHYSICAL TEMPERATURE AND DETUNING LIMIT

Fix $\beta, c > 0$, let $b = \beta c$, and take

$$\begin{aligned} &\beta_h = \beta(1-\varepsilon), \qquad \beta_c = \beta(1+\varepsilon), \\ &\frac{a}{c} = 1 + \frac{2\delta\varepsilon}{1-\varepsilon}, \quad 0 < \varepsilon < 1. \end{aligned} \quad \text{(S11)}$$

Here $\varepsilon := (T_h - T_c)/(T_h + T_c)$ is the relative temperature bias, and $\delta := (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$ is the dimensionless normalized detuning, fixed as $\varepsilon \downarrow 0$, and $a > 0$ for sufficiently small ε. It is zero at equal frequencies and one at equal occupations; positive swap work requires $0 < \delta < 1$. The exponents in Eq. (S1) are exactly

$$x = b[1 + (2\delta - 1)\varepsilon], \qquad y = b(1+\varepsilon). \quad \text{(S12)}$$

This preserves the stated physical temperatures while varying frequency. Holding x and y fixed instead would be a different family.

Write $m(\delta) = \max\{\delta, 1-\delta\} = \max\{|\delta|, |\delta - 1|\}$. The two ratios in Eq. (S7) yield

$$K = \frac{1}{2m(\delta)\varepsilon} + O(1), \qquad q = b + O(\varepsilon). \quad \text{(S13)}$$

Using Eq. (S10) gives

$$0 \le \mathcal{W} - W_{\mathrm{G}} = O_{b,c,\delta}\left(\varepsilon^{-2} e^{-b/[2m(\delta)\varepsilon]}\right). \quad \text{(S14)}$$

The absolute upper estimate can be taken uniform over fixed compact sets of δ, with b, c bounded away from zero and infinity and parameters positive. A relative estimate obtained by dividing by W_{G} is not uniform at its zeros.

In $0 < \delta < 1$, the exact swap work is

$$W_{\mathrm{G}} = \frac{2c\delta\varepsilon}{1-\varepsilon}\{r(b[1 + (2\delta - 1)\varepsilon]) - r(b(1+\varepsilon))\}. \quad \text{(S15)}$$

Since $r'(b) = -r(b)[r(b) + 1]$, at fixed interior δ,

$$\mathcal{W} = 4bc\, r(b)[r(b)+1]\delta(1-\delta)\varepsilon^2 + O(\varepsilon^3). \quad \text{(S16)}$$

Equation (S14) is stronger than the displayed power-series remainder: the entire difference between the exact swap work and unrestricted work lies beyond all algebraic orders. No exact exponential rate for that difference inside the window is claimed.

Matching rates when Gaussian work vanishes

For $\delta \le 0$ or $\delta \ge 1$, Eq. (S2) vanishes. Equation (S14) gives the upper logarithmic rate. To prove the lower rate, use a two-state swap between $|u, 0\rangle$ and $|0, v\rangle$ with

$$\begin{aligned} &\Delta E := au - cv > 0, \quad A := yv - xu > 0, \\ &W_{u,v} = C\Delta E\, e^{-yv}(e^A - 1). \end{aligned} \quad \text{(S17)}$$

This is a lower bound on the work optimized over all unitaries.

For fixed $\delta \le 0$, choose

$$u = \left\lfloor \frac{1+\varepsilon}{2(1-\delta)\varepsilon} \right\rfloor + 2, \qquad v = u - 1. \quad \text{(S18)}$$

Then $\Delta E/c \to 1/(1-\delta) > 0$ and

$$2b(1-\delta)\varepsilon < A \le 4b(1-\delta)\varepsilon. \quad \text{(S19)}$$

Thus its positive prefactor is bounded below by a constant times ε, while $yv = b/[2(1-\delta)\varepsilon] + O(1)$. For fixed $\delta \ge 1$, write $\eta = 2\delta\varepsilon/(1-\varepsilon)$ and take

$$u = \lfloor 1/\eta \rfloor + 2, \qquad v = u + 1. \quad \text{(S20)}$$

Here $\eta < \Delta E/c \le 2\eta$ and $A = b(1+\varepsilon) - 2b(\delta - 1)\varepsilon u \to b/\delta > 0$. Again the prefactor is bounded below by a constant times ε, with $yv = b/(2\delta\varepsilon) + O(1)$. The finite margins in these choices prevent arbitrarily close ties from defeating the lower bound. Together with Eq. (S14),

$$\ln \frac{\mathcal{W}}{c} = -\frac{b}{2m(\delta)\varepsilon} + O[\ln(1/\varepsilon)], \qquad \delta \notin (0,1). \quad \text{(S21)}$$

This includes $\delta = 0$ (equal frequencies) and $\delta = 1$ (equal occupation distributions at unequal frequencies). In both cases the logarithmic coefficient is $-b/2$ after multiplication by ε. It is not a prefactor asymptotic.

ENTROPY BOUND AND EXACT SATURATION

For the actual Hamiltonian at each ε, let γ_{β_*} have the same entropy as ρ. There is a unique $\beta_* > 0$, since

$$g[r(\beta_* a)] + g[r(\beta_* c)] = g[r(x)] + g[r(y)] \tag{S22}$$

has a strictly decreasing left side, from infinity to zero. The entropy bound is

$$\mathcal{B} = ar(x) + cr(y) - ar(\beta_* a) - cr(\beta_* c). \tag{S23}$$

Relative entropy gives $D(\rho\|\gamma_{\beta_*}) = \beta_* \mathcal{B}$. It also proves that every state of the same entropy has energy at least that of γ_{β_*} [8, 9].

For completeness, perform the expansion first at a fixed Hamiltonian. Let the two inverse temperatures be $\beta + \Delta\beta_j$, $j = h, c$, and let $V_j = \mathrm{Var}_{\gamma_\beta}(H_j)$. Write $\beta_* = \beta + \Delta\beta_*$. Since the derivative of Gibbs entropy is $-\beta V_j$, first-order entropy matching gives

$$\Delta\beta_* = \overline{\Delta\beta} + O(\Delta\beta^2), \quad \overline{\Delta\beta} = \frac{\sum_j V_j \Delta\beta_j}{\sum_j V_j}. \tag{S24}$$

The quadratic expansion of relative entropy between nearby Gibbs states then yields

$$\mathcal{B} = \frac{1}{2\beta} \sum_j V_j (\Delta\beta_j - \overline{\Delta\beta})^2 + O(\Delta\beta^3). \tag{S25}$$

For Eq. (S11), $\Delta\beta_h = -\beta\varepsilon$, $\Delta\beta_c = \beta\varepsilon$, and both variances are $c^2 r(b)[r(b)+1] + O(\varepsilon)$. Analyticity in the positive parameters makes this expansion valid along the varying-Hamiltonian family. Thus

$$\mathcal{B} = bc\, r(b)[r(b)+1]\varepsilon^2 + O(\varepsilon^3). \tag{S26}$$

Combining Eqs. (S16), (S21), and (S26) gives the accessible-work parabola in the Letter.

For arbitrary $0 < \beta_h < \beta_c$, choose $a/c = \sqrt{\beta_c/\beta_h}$. The swapped exponents are $\beta_c c$ on the mode of energy a, and $\beta_h a$ on the mode of energy c. Both correspond to physical inverse temperature $\sqrt{\beta_h \beta_c}$. The output is Gibbs, is unitarily connected to the input, and has its entropy. Hence $\mathcal{W} = W_{\mathrm{G}} = \mathcal{B}$ exactly. In Eq. (S11), this occurs at

$$\delta_*(\varepsilon) = \frac{\sqrt{1-\varepsilon^2} - (1-\varepsilon)}{2\varepsilon} = \frac{1}{2} - \frac{\varepsilon}{4} + O(\varepsilon^3). \tag{S27}$$

This exact check is not a claim of global optimization over frequency choices, which change the input energy and the Hamiltonian.

OCCUPATION SUPPORT AS AN OPERATIONAL RESOURCE

Let P_M project onto $m+n \le M$, with M a nonnegative integer. Restricted controls have $U = U_M \oplus I$ in $P_M\mathcal{H} \oplus (I - P_M)\mathcal{H}$. The state is not truncated or renormalized. Let $\mathcal{W}_M$ be the maximum work in this class.

For physical inverse temperatures $\beta(1 \mp \varepsilon)$, a higher-energy state has a higher population exactly when its occupation differences satisfy

$$0 < a\Delta m + c\Delta n < \varepsilon(a\Delta m - c\Delta n). \tag{S28}$$

The two inequalities require $\Delta m > 0$ and $\Delta n < 0$. Put $u = \Delta m$ and $v = -\Delta n$. Any inverted pair inside the cutoff yields integers $1 \le u, v \le M$ such that

$$0 < au - cv < \varepsilon(au + cv). \tag{S29}$$

Conversely these integers give the pair $|u, 0\rangle, |0, v\rangle$, also inside the cutoff. Since population ordering characterizes passivity in finite dimension, Eq. (S29) is necessary and sufficient for $\mathcal{W}_M > 0$; superpositions do not alter it.

For $0 < \delta < 1$, $u = v = 1$ satisfies the criterion, so $M_{\min} = 1$. For fixed $\delta \le 0$, an inversion must have $u = v + k$ with $k \ge 1$ and $u > (1+\varepsilon)k/[2(1-\delta)\varepsilon]$. The first $k = 1$ pair also satisfies the energy inequality for sufficiently small ε, giving

$$M_{\min} = \left\lfloor \frac{1+\varepsilon}{2(1-\delta)\varepsilon} \right\rfloor + 1. \tag{S30}$$

At $\delta = 0$ this is exactly $\lfloor (1-\varepsilon)/(2\varepsilon) \rfloor + 2$. For fixed $\delta \ge 1$, write $v = u + k$. Energy inversion requires $u > k(1-\varepsilon)/(2\delta\varepsilon)$; the first $k = 1$ pair has the required population order for sufficiently small ε, so

$$M_{\min} = \left\lfloor \frac{1-\varepsilon}{2\delta\varepsilon} \right\rfloor + 2. \tag{S31}$$

The latter two asymptotics are $\Theta(\varepsilon^{-1})$. Equalities producing zero work do not count as activation.

Now fix M and an interior δ. For all sufficiently small ε, distinct shells through M are ordered both in energy and in population. Within each shell, swapping m, n reverses the initial population order and is the passive rearrangement. It is therefore the exact restricted optimum. Its work is

$$\mathcal{W}_M = (a-c)C \sum_{m+n \le M} (m-n) e^{-xm-yn}. \tag{S32}$$

At $\varepsilon = 0$ the odd sum vanishes. Differentiate the population factor using Eq. (S12), and use

$$\sum_{m=0}^{N} (2m - N)m = \frac{N(N+1)(N+2)}{6}. \tag{S33}$$

With $z = e^{-b}$ this gives

$$\begin{aligned} \mathcal{W}_M &= 4bc\,\delta(1-\delta) R_M \varepsilon^2 + O(\varepsilon^3), \\ R_M &= (1-z)^2 \sum_{N=1}^{M} \binom{N+2}{3} z^N. \end{aligned} \tag{S34}$$

The generating function gives $R_\infty = z/(1-z)^2 = r(b)[r(b)+1]$, proving the limiting fraction in the Letter. It is the cumulative distribution function of a negative-binomial series, since $(1-z)^4 \sum_{N\geq 1} \binom{N+2}{3} z^{N-1} = 1$. Thus it converges exponentially with a cubic prefactor in M. Every target fraction below one has a finite cutoff independent of ε. This statement concerns support, not a lower bound on duration, gate count, measurement samples, or control energy.

GAUSSIAN INPUTS AND MEASUREMENT INTERPRETATION

For any finite-energy state ρ and a Gaussian unitary U_G attaining its Gaussian minimum, let $\sigma = U_G \rho U_G^\dagger$. Unitary invariance of E_P gives

$$\begin{aligned} \mathcal{W}(\rho) - W_{\mathrm{G}}(\rho) &= \mathrm{tr}(H\sigma) - E_P(\rho) \\ &= \mathcal{W}(\sigma). \end{aligned} \tag{S35}$$

For a full-rank Gaussian input with finite covariance and a positive two-mode oscillator Hamiltonian, a minimizing σ can be chosen as a thermal product with the larger occupation on the lower frequency [2, 3]. Its two finite positive thermal exponents are determined by the symplectic eigenvalues of the covariance. Equation (S9) applies with $W_{\mathrm{G}}(\sigma) = 0$, giving $W_{\mathrm{G}}(\rho) \leq \mathcal{W}(\rho) \leq W_{\mathrm{G}}(\rho) + e_+ F$. The finite-temperature theorem does not cover a pure Williamson mode ($x = \infty$ or $y = \infty$) without a separate limiting analysis.

This bounds the extra work under the stated preparation model. Arbitrary non-Gaussian states sharing the same first and second moments need not have thermal-product Gaussian-passive normal forms, so those moments alone do not establish the bound. A proposed occupation experiment must independently justify the thermal-product or Gaussian preparation model. Frequency calibration and occupation means determine x, y; statistical uncertainty must be propagated through the finite bound, rather than evaluated only at fitted central values. The mean energy decrease of a known swap is measurable without reconstructing the unrestricted passive state. No experimental data, sample-complexity lower bound, or optimization of control power is reported here.

SPECTRAL NUMERICS AND REPRODUCIBILITY

The numerical calculation uses $\varepsilon = 1/j$ for $j = 20, 40, 80, 160$, $\delta = -1/2, 0, 1/4, 1/2, 3/4, 1, 3/2$, and $b = 0.5, 1, 2$, totaling 84 cases. It checks the entire residual after the Gaussian-optimal preliminary swap, rather than subtracting two nearly equal floating-point energies. The stored program is `code/detuning_sort.py`; its JSON output and execution record are in `data/pc2_results.json` and `data/pc2_execution.json`. The computation used one CPU process and Python's standard library. Its independently transferred input and output hashes were verified before import.

For $\delta = p/q_\delta$ in lowest terms, energies in units of c have integer keys

$$E_{mn}/c = \frac{[q_\delta(j-1) + 2p]m + q_\delta(j-1)n}{q_\delta(j-1)}. \tag{S36}$$

The exponents $xm + yn$ also have integer keys, with denominator $q_\delta j$ and common factor b. After the Gaussian swap, their two coefficients are exchanged when required. Sorting integers handles exact degeneracies and near-ties without numerical comparison errors.

We retain *every* state with physical energy $E_{mn} \leq cQ$, including all ties. This is a genuine prefix of the energy spectrum. A total-number triangle would not generally be such a prefix at detuned frequencies. Within the prefix, states with each equal probability key are grouped, and their total initial energy is compared with the total assigned passive energy. Only nonzero group differences are evaluated numerically. This preserves cancellations in the already-passive part exactly.

Let A be the positive, unnormalized retained part of the Gaussian-passive state σ, and $T = \sigma - A \geq 0$. Sorting A inside its energy prefix realizes its unrestricted passive energy; denote the resulting work by W_Q. Then

$$W_Q \leq \mathcal{W}(\sigma) \leq W_Q + \mathrm{tr}(HT). \tag{S37}$$

The lower bound extends the prefix unitary by identity. For the upper bound, $E_P(A+T) \geq E_P(A) + E_P(T) \geq E_P(A)$: the infimum of a sum is no smaller than the sum of infima. Subtracting from the initial energy proves the claim. Normalization of A is unnecessary.

If $E > cQ$, then $N > cQ/e_+$. Therefore, with $L = \lfloor cQ/e_+ \rfloor + 1$,

$$\mathrm{tr}(HT) \leq e_+ F(L; e^{-q}, C). \tag{S38}$$

This is a rigorous analytic truncation bound. Group sums are evaluated with Decimal at 90 and 120 significant digits. Their agreement is a numerical stability check, not a directed-rounding proof. Stored decimal values should not be represented as formally rounded interval endpoints. All finite-remainder comparisons use high precision with large numerical margins. The largest retained spectrum contained 65 821 states. All 84 residual-plus-tail estimates lay below Eq. (S9), and all total-work-plus-tail estimates lay below $\mathcal{B}$. The largest discarded-tail bound relative to total work was 1.656×10^{-37}; the largest relative change of a nonzero residual between 90 and 120 digits was 5.82×10^{-88}. The tail need not be small relative to the much tinier non-Gaussian correction:

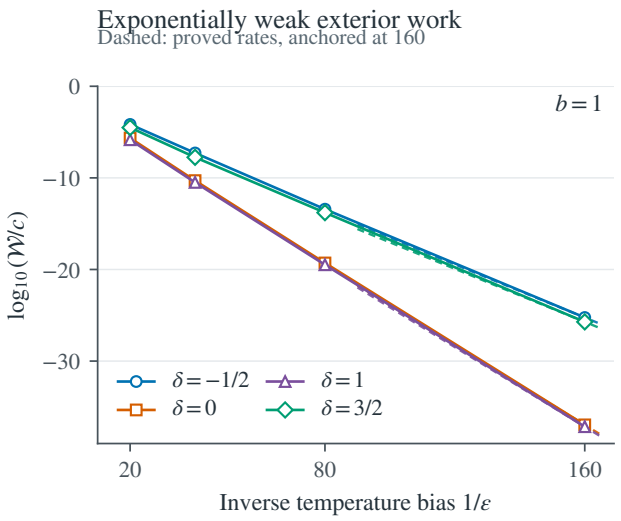


FIG. S1. Exponentially weak work at and outside the swap window, at $b = \beta\hbar\omega_c = 1$. Here $c = \hbar\omega_c$, $\varepsilon = (T_h - T_c)/(T_h + T_c)$, and $\delta = (\omega_h/\omega_c - 1)/(T_h/T_c - 1)$. Symbols show numerical lower estimates of $\mathcal{W}/c$ from passive spectral sorting for $\delta = -1/2, 0, 1, 3/2$; solid segments connect the points to guide the eye. Dashed guides have the proved asymptotic slopes in Eq. (S21), anchored at $1/\varepsilon = 160$; they are not fits. The pairs $\delta = 0, 1$ and $\delta = -1/2, 3/2$ have respective limiting slopes $-1/(2\ln 10)$ and $-1/(3\ln 10)$ on these axes. Analytic omitted-tail contributions are smaller than the symbols. Precision agreement is a floating-point stability check, not a directed-rounding proof.

its largest ratio to that correction was 5.50, near $\delta = 1/2$. Thus the calculation tightly resolves total work but does not claim relative resolution of every residual. The original stored total-work field used 28-digit addition; plotted and checked totals are reconstructed from the 120-digit components. The distributed main script corrects that serialization issue, and a full repeated run reproduces the component values against the original PC2 output.

Figures are generated by `code/plot_detuning.py`. Entropy matching is solved as a monotone scalar equation at high precision. Curves based only on Eq. (S2) are labeled as swap work; spectral markers show the full sorted work within the stated analytic truncation control. No fitted rate is used as evidence for the theorem. The clean source package builds with REVTeX 4.2; the full reproducibility package includes raw numerical outputs, code, source notes, and numerical checks.

SCOPE OF THE LIMIT

All logarithmic rates concern fixed b, c, δ as $\varepsilon \downarrow 0$ in Eq. (S11). They do not describe fixed nonzero frequency detuning, for which spectral arithmetic may be relevant. Even near an edge, a leading linear approximation to the frequency ratio can change the answer: $a/c = 1 + 2\varepsilon$ corresponds to $\delta = 1 - \varepsilon$ and produces cubic swap work; the exact $\delta = 1$ path has exponentially small unrestricted work. The finite bound remains applicable to both, whereas the fixed-δ relative limit cannot be substituted without checking the path.

The theorem concerns two modes, or a two-mode Gaussian orbit under the stated conditions. A limit in which the number of modes grows is not included. Frequency choice compares different preparations and devices; work required to tune a frequency after preparation is outside the fixed-Hamiltonian extraction stroke and cannot be ignored in a complete cycle. These boundaries are essential to the physical interpretation, rather than numerical limitations.

[1] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[2] E. G. Brown, N. Friis, and M. Huber, Passivity and practical work extraction using Gaussian operations, New J. Phys. **18**, 113028 (2016).

[3] F. E. Quintela Rodriguez, F. A. Mele, S. F. E. Oliviero, V. Giovannetti, L. Lami, and V. Cavina, Extracting energy via bosonic Gaussian operations, arXiv:2503.21748 (2025).

[4] M. F. Sacchi, Thermodynamic uncertainty relations for bosonic Otto engines, Phys. Rev. E **103**, 012111 (2021).

[5] W. Pusz and S. L. Woronowicz, Passive states and KMS states for general quantum systems, Commun. Math. Phys. **58**, 273 (1978).

[6] A. Lenard, Thermodynamical proof of the Gibbs formula for elementary quantum systems, J. Stat. Phys. **19**, 575 (1978).

[7] A. E. Allahverdyan, R. Balian, and Th. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, Europhys. Lett. **67**, 565 (2004).

[8] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, Phys. Rev. E **87**, 042123 (2013).

[9] A. Sone and S. Deffner, Quantum and classical ergotropy from relative entropies, Entropy **23**, 1107 (2021).